Normal distⁿ

Discovered by De-Moivre (1733)

A random variable X is said to have a Normal distⁿ with parameters μ and σ if its pdf is given by

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty$$

where $-\infty < \mu < \infty$, $\sigma > 0$

~~Parameters~~ The parameters μ is called mean and σ^2 is called variance of that p.d.f.

Note:

1. If X has ~~follows~~ the above defined pdf, then it can be written as $X \sim N(\mu, \sigma^2)$
2. If $\mu = 0$ and $\sigma^2 = 1$ then $X \sim N(0,1)$ which is called Standard Normal distribution

Moments of Normal distⁿ.

odd order central moments

$$\mu_{2r+1} = \int_{-\infty}^{\infty} (x-\mu)^{2r+1} f(x;\mu,\sigma)\,dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Let $\frac{x-\mu}{\sigma} = z$, then $dx = \sigma\,dz$

$$\therefore \mu_{2r+1} = \int_{-\infty}^{\infty} (\sigma z)^{2r+1} \cdot \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}z^2} \cdot \cancel{\sigma}\,dz$$

$$= \int_{-\infty}^{\infty} \sigma^{2r+1} z^{2r+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$\therefore \quad \mu_{2r+1} = \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2} dz$$

$$= 0$$

Since the integrand $z^{2r+1} e^{-z^2/2}$ is an odd function of z

So all odd order central moments of Normal distn is zero. i.e. $\mu_1 = 0$ (always.)

$$\mu_3 = \mu_5 = \mu_7 \cdots = \mu_{2r+1} = 0$$

Even order central moments.

$$\mu_{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x;\mu,\sigma)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz \qquad \left(\text{Putting } z = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \left(2\int_{0}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz \right)$$

Let $\frac{z^2}{2} = t$

$\therefore z\,dz = dt$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \left(2 \cdot \int_{0}^{\infty} (2t)^r e^{-t} \frac{dt}{\sqrt{2t}} \right)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \left(\int_{0}^{\infty} t^{\left(r - \frac{1}{2}\right)} e^{-t} dt \right)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \left(\int_{0}^{\infty} e^{-t} t^{\left(r+\frac{1}{2}\right)-1} dt \right)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \cdot \Gamma\left(r + \frac{1}{2}\right)$$

$$= \frac{\sigma^{2n}}{\sqrt{\pi}} \cdot 2^n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots \frac{1}{2}\Gamma\tfrac{1}{2}$$

$$\left[\text{as } \Gamma(n) = (n-1)\Gamma(n-1)\right]$$

$$= \frac{\sigma^{2n}}{\sqrt{\pi}} 2^n \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{1}{2} \cdot \sqrt{\pi}$$

$$= \sigma^{2n}(2n-1)(2n-3)\cdots 3.1$$

$\therefore \mu_2 = \mu_{2.1} = \sigma^2$

$\mu_4 = \mu_{2.2} = 3\sigma^4$

So, skewness $= 0$ i.e. symmetric.

kurtosis $= \dfrac{\mu_4}{\mu_2^2} = 3$ i.e. mesokurtic

Note:

- If $X_1 \sim N(\mu_1, \sigma_1^2)$
  $X_2 \sim N(\mu_2, \sigma_2^2)$ independently,

  then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

  $X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

- in general,

  If $X_i \sim N(\mu_i, \sigma_i^2)$ $i = 1(1)n$

  $$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

- If all $a_i = 1/n$, ~~then~~ $\mu_i = \mu$ and $\sigma_i = \sigma$,

  then $\sum_{i=1}^{n} a_i X_i = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \sim N\left(\frac{1}{n}\sum_{i=1}^{n}\mu, \frac{1}{n^2}\sum \sigma^2\right)$

  ie $\bar{X} \sim N(\mu, \sigma^2/n)$

~~Mode of Normal distri~~

Properties:

i) The curve of Normal distⁿ pdf is bell shaped and symmetrical about $x = \mu$.

ii) Mean, Median and mode of the distⁿ coincide.

iii) $\beta_1 = 0$, $\beta_2 = 3$

iv) $\mu_{2r+1} = 0$, $\mu_{2r} = 1.3.5 \ldots (2r-1)\sigma^{2r}$

v) The points of inflexion of the curve are

$x = \mu \pm \sigma$, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

Point of inflexion is that point where first order derivative and second order derivative is zero.

i.e. $\frac{d f(x; \mu, \sigma)}{dx} = 0$ and $\frac{d^2 f(x; \mu, \sigma)}{dx^2} = 0$

(do yourself)

vi) The pdf of standard Normal distⁿ is given by,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty.$$

with the corresponding distribution function,

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(u)\,du$$

$$= \cancel{\int_{-\infty}^{z} \phi(z)\,dz}$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

Note that

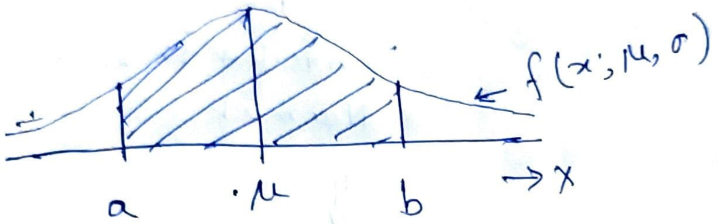
$$\Phi(-z) = 1 - \Phi(z), \quad z > 0$$

and $P(a < X < b)$

$$= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$ where $X \sim N(\mu, \sigma^2)$

$$= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$



$P(a < x < b)$ means the shaded area.

So to compute the area between two lines $x = a$ and $x = b$ under the curve you have to do the following,

$$\int_a^b f(x; \mu, \sigma)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Let $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma\, dz$.

So

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz$$ (looks like standard normal form)

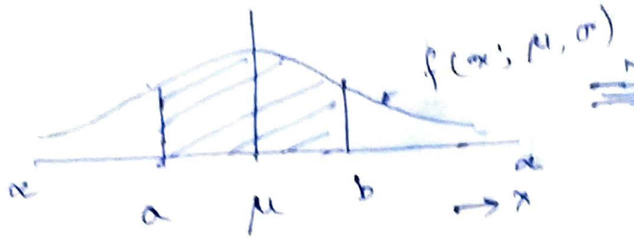
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\frac{b-\mu}{\sigma}} e^{-1/2\, z^2} dz - \int_{-\infty}^{\frac{a-\mu}{\sigma}} e^{-1/2\, z^2} dz \right]$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Page ⑥ So, to calculate the $P(a < X < b)$

where $X \sim N(\mu, \sigma^2)$

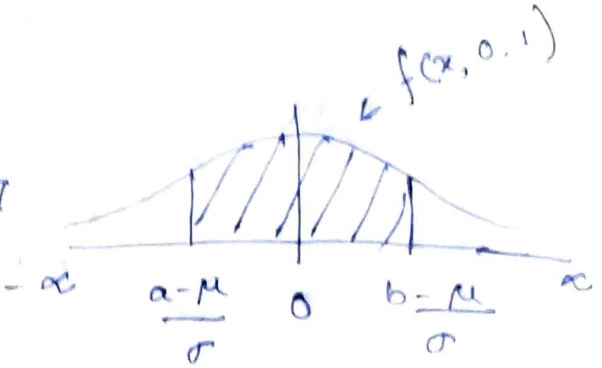
we follow the steps.



$\underbrace{\phantom{a<x<b}}$

$P(a < x < b)$

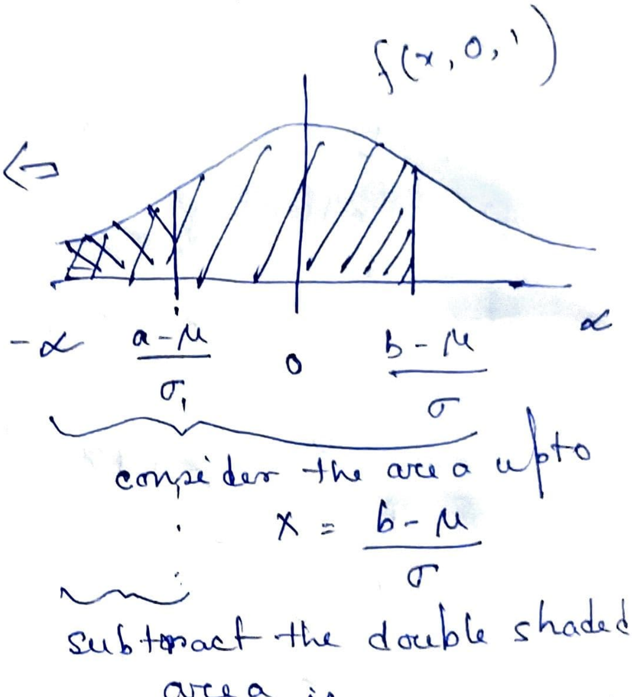
$\Rightarrow$



$P\left(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}\right)$

i.e converted to standard Normal

$\Downarrow$



i.e.

$$\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \Leftarrow$$

$$= \Phi(b') - \Phi(a')$$

consider the area upto

$x = \frac{b-\mu}{\sigma}$

subtract the double shaded area i.e.

$x = -\infty$ to $\frac{a-\mu}{\sigma}$

See

I have taken

$a < 0$

and $b > 0$

if $a < 0$, $b < 0$

or $a > 0$, $b > 0$

do the same

Now the point is how to get the values of $\Phi(b')$ or $\Phi(a')$

- to do this go the page 600, ~~Fund~~ Fundamentals of Statistics vol. 1 (standard Normal table)

this table has three columns.

$z \mid \phi(z) \mid \Phi(z)$

z means $\rightarrow$ X values.

for each X i.e. z you get $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

and $\Phi(z)$ i.e. $\int_{-\alpha}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \int_{-\alpha}^{z} \phi(z)\, dz$

which is the area upto the point $X = z$. from $-\alpha$

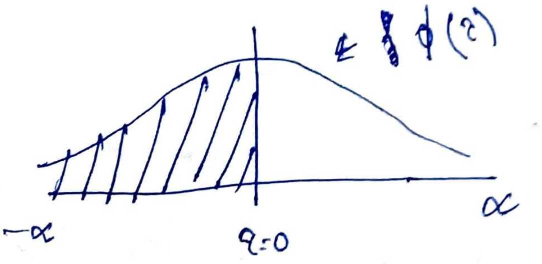
Here z is considered as positive value.

See the first value, $z = 0.00$

$\phi(z) = 0.3989423$

$\Phi(z) = 0.5$

that means



$\Phi(0.00) = 0.5$ as it is ~~pdf~~ Normal dist^n pdf, so the total area under the ~~fi~~ curve is 1

i.e. $\int_{-\alpha}^{\alpha} \phi(z)\, dz = 1$

and it is symmetric dist^n so $\int_{-\alpha}^{0} \phi(z)\, dz = 0.5$

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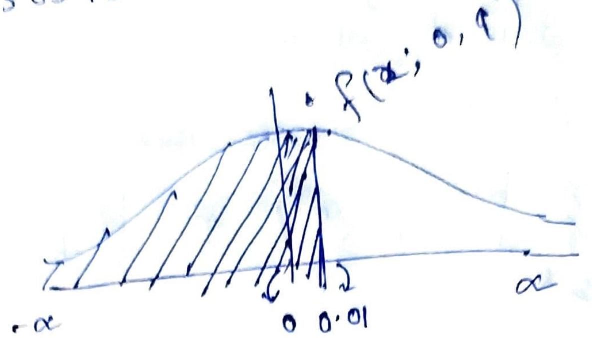
See the Second value

$x = 0.01$

$\phi(x) = 0.3989223$

$\Phi(x) = 0.5039894$

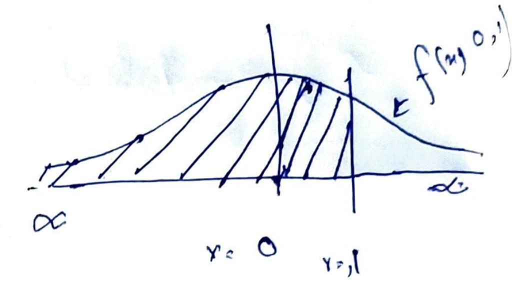
that means,



$\Phi(0.01) = 0.5039894.$

i.e. $\int_{-\infty}^{0.01} \phi(x)\, dx = 0.5039894$

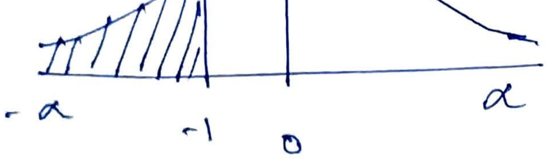
See if $\begin{cases} x = 1.00 \\ \Phi(x) = 0.8413447 \end{cases}$



the shaded area $= 0.8413447$

Now what if $\{ x = -1.00$

i.e. what is the area under the ~~curve~~ shaded area?

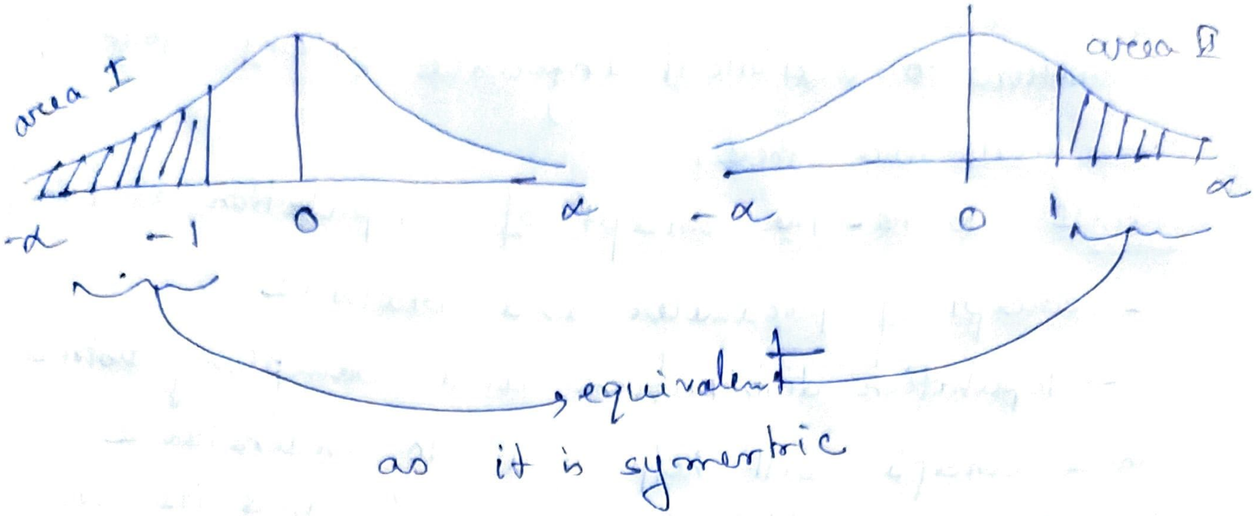


it is $\Phi(-1) = 1 - \Phi(1)$ (see property vi)

$= 1 - 0.8413447$

$=$ calculate the value

For better understanding,



area I

$= $ area II

$$= \int_{1}^{\alpha} \phi(x)\, dx$$

$$= \int_{-\alpha}^{\alpha} \phi(x)\, dx - \int_{-\alpha}^{1} \phi(x)\, dx$$

$$= 1 - \Phi(1) = 1 - 0.8413447$$