We shall now obtain the relationship between the Einstein A coefficient and the spontaneous lifetime of level 2. Let us assume that an atom in level 2 can make a spontaneous transition only to level 1. Then since the number of atoms making spontaneous transitions per unit time per unit volume is $A_{21}N_2$, we may write the rate of change of population of level 2 with time due to spontaneous emission as

$$\frac{dN_2}{dt} = -A_{21}N_2 \tag{4.11}$$

the solution of which is

$$N_2(t) = N_2(0)e^{-A_{21}t} \tag{4.12}$$

Thus the population of level 2 reduces by 1/e in a time $t_{sp} = 1/A_{21}$ which is called the spontaneous lifetime associated with the transition 2→1.

If the group of atoms is in thermal equilibrium, it can be shown from thermodynamics that the ratio of the number of atoms in each state is given by a Boltzmann distribution:

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT}$$

where T is the thermodynamic temperature of the group of atoms, and k is Boltzmann's constant.

We may calculate the ratio of the populations of the two states at room temperature ($T \approx 300\ K$) for an energy difference ΔE that corresponds to light of a frequency corresponding to visible light ($f \approx 5 \times 10^{14}$ Hz). In this case $\Delta E = E_2 - E_1 \approx 2.07$ eV, and $kT \approx 0.026$ eV. Since $E_2 - E_1 \gg kT$, it follows that the argument of the exponential in the equation above is a large negative number, and as such N_2/N_1 is vanishingly small; that is, there are almost no atoms in the excited state. When in thermal equilibrium, then it is seen that the lower energy state is more populated than the higher energy state, and this is the normal state of the system. As T increases, the number of electrons in the high-energy state (N_2) increases, but N_2 never exceeds N_1 for a system at thermal equilibrium; rather, at infinite temperature, the populations N_2 and N_1 become equal. In other words, a population inversion ($N_2/N_1 > 1$) can never exist for a system at thermal equilibrium. To achieve population inversion therefore requires pushing the system into a non-equilibrated state.

E.2 The Interaction of Light with Matter

There are three types of possible interactions between a system of atoms and light that are of interest:

E.2.1 Absorption

If light photons of frequency f_{12} pass through the group of atoms, there is a possibility of the light being absorbed by atoms which are in the ground state, which will cause them to be excited to the higher energy state. The probability of absorption is proportional to the radiation intensity of the light, and also to the number of atoms currently in the ground state, N_1.

E.2.2 *Spontaneous Emission*

If a collection of atoms are in the excited state, spontaneous decay events to the ground state will occur at a rate proportional to N_2, the number of atoms in the excited state. The energy difference between the two states ΔE_{21} is emitted from the atom as a photon of frequency f_{21} as given by the frequency–energy relation above.

The photons are emitted stochastically, and there is no fixed phase relationship between photons emitted from a group of excited atoms; in other words, spontaneous emission is incoherent. In the absence of other processes, the number of atoms in the excited state at time t, is given by

$$N_2(t) = N_2(0)\,\mathrm{e}^{\frac{-t}{\tau_{21}}}$$

where $N_2(0)$ is the number of excited atoms at time $t = 0$, and τ_{21} is the lifetime of the transition between the two states.

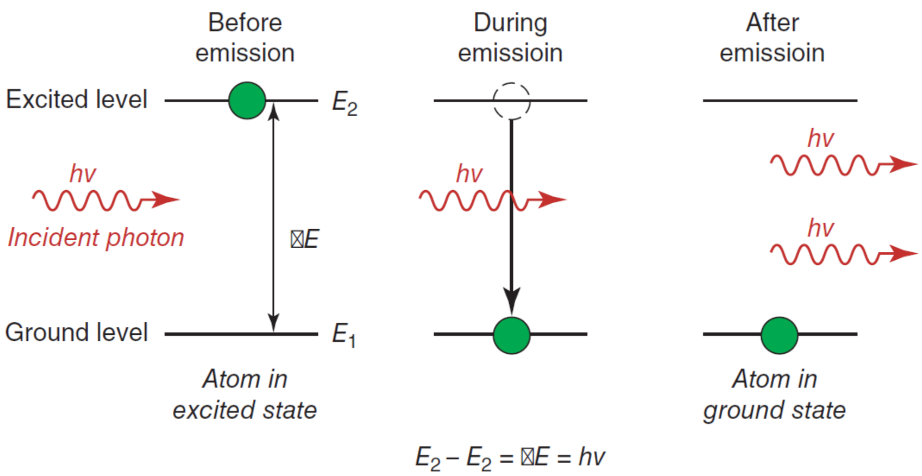

Figure E.1 Illustrating stimulated emission.

E.2.3 *Stimulated Emission*

If an atom is already in the excited state, it may be perturbed by the passage of a photon that has a frequency f_{21} corresponding to the energy gap ΔE of the excited state to ground state transition. In this case, the excited atom relaxes to the ground state, and is induced to produce a second photon of frequency f_{21}. The original photon is not absorbed by the atom, and so the result is two photons of the same frequency. This process is known as *stimulated emission* and is portrayed in Figure E.1.

Specifically, an excited atom will act like a small electric dipole which will oscillate with the external field provided. One of the consequences of this oscillation is that it encourages electrons to decay to the lowest energy state. When this happens due to the presence of the electromagnetic field from a photon, a photon is released in the same phase and direction as the ‘stimulating’ photon, and is called stimulated emission.

The rate at which stimulated emission occurs is proportional to the number of atoms N_2 in the excited state, and the radiation density of the light. The base probability of a photon causing stimulated emission in a single excited atom was shown by Albert Einstein to be exactly equal to the probability of a photon being absorbed by an atom in the ground state. Therefore, when the numbers of atoms in the ground and excited states are equal, the rate of stimulated emission is equal to the rate of absorption for a given radiation density.

The critical detail of stimulated emission is that the induced photon has the same frequency and phase as the incident photon. In other words, the two photons are coherent. It is this property that allows optical amplification, and the production of a laser system. During the operation of a laser, all three light-matter interactions described above are taking place. Initially, atoms are energized from the ground state to the excited state by a process called pumping, described below. Some of these atoms decay via spontaneous emission, releasing incoherent light as photons of frequency, ν. These photons are fed back into the laser medium, usually by an optical resonator. Some of these photons are absorbed by the atoms in the ground state and the photons are lost to the laser process. However, some photons cause stimulated emission in excited-state atoms, releasing another coherent photon. In effect, this results in optical amplification.

If the number of photons being amplified per unit time is greater than the number of photons being absorbed, then the net result is a continuously increasing number of photons being produced; the laser medium is said to have a gain of greater than unity.

Recall from the descriptions of absorption and stimulated emission above that the rates of these two processes are proportional to the number of atoms in the ground and excited states, N_1 and N_2, respectively. If the ground state has a higher population than the excited state ($N_1 > N_2$), the process of absorption dominates and there is a net attenuation of photons. If the populations of the two states are the same ($N_1 = N_2$), the rate of absorption of light exactly balances the rate of emission; the medium is then said to be optically transparent.

If the higher energy state has a greater population than the lower energy state ($N_1 < N_2$), then the emission process dominates, and light in the system undergoes a net increase in intensity. It is thus clear that to produce a faster rate of stimulated emissions than absorptions, it is required that the ratio of the populations of the two states is such that $N_2/N_1 > 1$. In other words, a population inversion is required for laser operation.

E.5 Three-Level Lasers

To achieve non-equilibrium conditions, an indirect method of populating the excited state must be used. To understand how this is done, we may use a slightly more realistic model, that of a *three-level laser*. Again consider a group of N atoms, this time with each atom able to exist in any of three energy states, levels 1, 2 and 3, with energies E_1, E_2 and E_3, and populations N_1, N_2 and N_3, respectively.

Note that $E_1 < E_2 < E_3$; that is, the energy of level 2 lies between that of the ground state and level 3.

Initially, the system of atoms is at thermal equilibrium, and the majority of the atoms will be in the ground state, that is, $N_1 \approx N$, $N_2 \approx N_3 \approx 0$. If we now subject the atoms to light of a frequency, $f_{13} = \frac{1}{h}(E_3 - E_1)$ the process of optical absorption will excite the atoms from the ground state to level 3. This process is called *pumping* and does not necessarily always directly involve light absorption; other methods of exciting the laser medium, such as electrical discharge or chemical reactions, may be used. The level 3 is sometimes referred to as the *pump level* or *pump band*, and the energy transition $E_1 \rightarrow E_3$ as the *pump transition*, which is shown as the arrow marked **P** in Figure E.2.

If we continue pumping the atoms, we will excite an appreciable number of them into level 3, such that $N_3 > 0$. In a medium suitable for laser operation, we require these excited atoms to quickly decay to level 2. The energy released in this transition may be emitted as a photon (spontaneous emission); however, in practice the 3→2 transition (labeled **R** in the diagram) is usually *radiationless*, with the energy being transferred to vibrational motion (heat) of the host material surrounding the atoms, without the generation of a photon.

An atom in level 2 may decay by spontaneous emission to the ground state, releasing a photon of frequency f_{12} (given by $E_2 - E_1 = hf_{12}$), which is shown as the transition **L**, called the *laser transition* in the diagram. If the lifetime of this transition, τ_{21} is much longer than the lifetime of the radiationless $3 \rightarrow 2$ transition τ_{32} (if $\tau_{21} \gg \tau_{32}$, known as a *favourable lifetime ratio*), the population of the E_3 will be essentially zero ($N_3 \approx 0$) and a population of excited state atoms will accumulate in level 2 ($N_2 > 0$). If over half the N atoms can be accumulated in this state, this will exceed the population of the ground state N_1. A population inversion ($N_2 > N_1$) has thus been achieved between level 1 and 2, and optical amplification at the frequency f_{21} can be obtained.

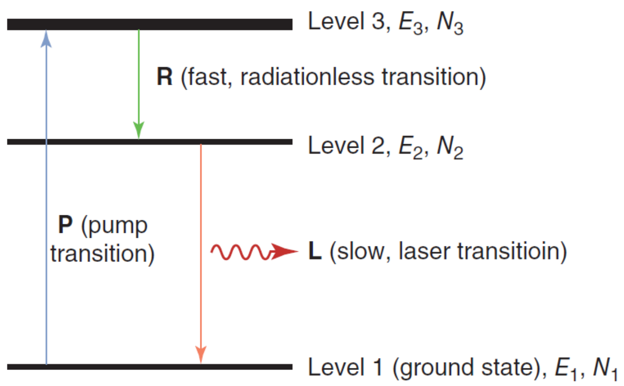

Figure E.2 A three-level laser energy diagram.

Because at least half the population of atoms must be excited from the ground state to obtain a population inversion, the laser medium must be very strongly pumped. This makes three-level lasers rather inefficient, despite being the first type of laser to be discovered (based on a ruby laser medium, by Theodore Maiman in 1960). A three-level system could also have a radiative transition between levels 3 and 2, and a non-radiative transition between levels 2 and 1. In this case, the pumping requirements are weaker. In practice, most lasers are *four-level lasers*, as described in the next section.

E.6 Four-Level Lasers

As illustrated in Figure E.3 there are four energy levels, energies E_1, E_2, E_3, E_4, and populations N_1, N_2, N_3, N_4, respectively. The energies of each level are such that $E_1 < E_2 < E_3 < E_4$.

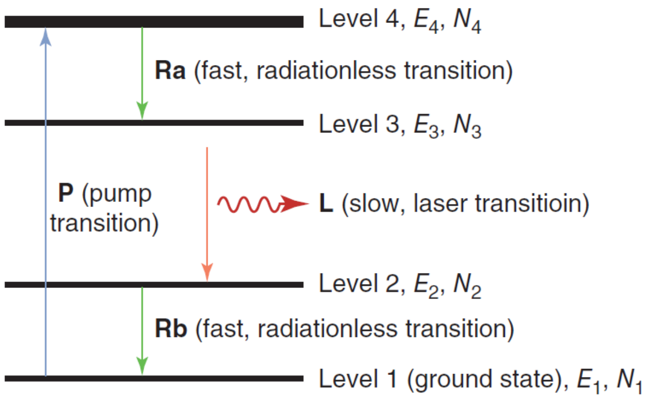

Figure E.3 A four-level laser energy diagram.

In this system, the pumping transition **P** excites the atoms in the ground state (level 1) into the pump band (level 4). From level 4, the atoms again decay by a fast, non-radiative transition **Ra** into the level 3. Since the lifetime of the laser transition **L** is long compared to that of **Ra** ($\tau_{32} \gg \tau_{43}$), a population accumulates in level 3 (the *upper laser level*), which may relax by spontaneous or stimulated emission into level 2 (the *lower laser level*). This level likewise has a fast, non-radiative decay **Rb** into the ground state.

As before, the presence of a fast, radiationless decay transitions results in the population of the pump band being quickly depleted ($N_4 \approx 0$). In a four-level system, any atom in the lower laser level E_2 is also quickly de-excited, leading to a negligible population in that state ($N_2 \approx 0$). This is important, since any appreciable population accumulating in level 3, the upper laser level, will form a population inversion with respect to level 2. That is, as long as $N_3 > 0$, then $N_3 > N_2$ and a population inversion is achieved. Thus optical amplification, and laser operation, can take place at a frequency of f_{32} ($E_3 - E_2 = hf_{32}$).

Since only a few atoms must be excited into the upper laser level to form a population inversion, a four-level laser is much more efficient than a three-level one, and most practical lasers are of this type. In reality, many more than four energy levels may be involved in the laser process, with complex excitation and relaxation processes involved between these levels. In particular, the pump band may consist of several distinct energy levels, or a continuum of levels, which allow optical pumping of the medium over a wide range of wavelengths.

Note that in both three- and four-level lasers, the energy of the pumping transition is greater than that of the laser transition. This means that, if the laser is optically pumped, the frequency of the pumping light must be greater than that of the resulting laser light. In other words, the pump wavelength is shorter than the laser wavelength. It is possible in some media to use multiple photon absorptions between multiple lower-energy transitions to reach the pump level; such lasers are called *up-conversion* lasers.

While in many lasers the laser process involves the transition of atoms between different electronic energy states, as described in the model above, this is not the only mechanism that can result in laser action. For example, there are many common lasers (e.g., dye lasers, carbon dioxide lasers) where the laser medium consists of complete molecules, and energy states correspond to vibrational and rotational modes of oscillation of the molecules. This is the case with water masers that occur in nature.

In some media it is possible, by imposing an additional optical or microwave field, to use quantum coherence effects to reduce the likelihood of an excited-state to ground-state transition. This technique, known as lasing without inversion, allows optical amplification to take place without producing a population inversion between the two states.

Laser Rate Equations

5.1 Introduction

In Chapter 4 we studied the interaction of radiation with matter and found that under the action of radiation of proper frequencies, the atomic populations of various energy levels change. In this chapter, we will be studying the rate equations which govern the rate at which populations of various energy levels change under the action of the pump and in the presence of laser radiation. The rate equations approach provides a convenient means of studying the time dependence of the atomic populations of various levels in the presence of radiation at frequencies corresponding to the different transitions of the atom. It also gives the steady-state population difference between the actual levels involved in the laser transition and allows one to study whether an inversion of population is achievable in a transition and, if so, what would be the minimum pumping rate required to maintain a steady population inversion between two levels, the gain that such a medium would provide at and near the transition frequency, and the phase shift effects that such a medium would introduce are discussed in detail in Chapter 6. Thus Chapter 6 discusses the behavior of a system having two levels when there is a population inversion between the two levels, and this chapter deals with the means of obtaining an inversion between two levels of an atomic system by making use of other energy levels. The rate equations can also be solved to obtain the transient behavior of the laser, which gives rise to phenomena like Q-switching and spiking.

The atomic rate equations along with the rate equation for the photon number in the cavity form a set of coupled nonlinear equations. These equations can be solved under the steady-state regime and one can study the evolution of the photon number as one passes through the threshold pumping region.

In Section 5.2 we discuss a two-level system and show that it is not possible to achieve population inversion in steady state in a two-level system. Sections 5.3 and 5.4 discuss three-level and four-level laser systems and obtain the dependence of inversion on the pump power. In Section 5.5 we obtain the variation of laser power around threshold showing the sudden increase in the output power as a function of pumping. This is a very characteristic behavior of a laser. Finally in Section 5.6 we discuss the optimum output coupling for maximizing the output power of a laser.

5.2 The Two-Level System

We first consider a two-level system consisting of energy levels E_1 and E_2 with N_1 and N_2 atoms per unit volume, respectively [see (Fig. 5.1)]. Let radiation at frequency ω with energy density u be incident on the system. The number of atoms per unit volume which absorbs the radiation and is excited to the upper level will be [see Eq. (4.18)]

$$\Gamma_{12} = \frac{\pi^2 c^3}{\hbar\omega^3 t_{sp} n_0^3} u g(\omega) N_1 = W_{12} N_1 \tag{5.1}$$

where

$$W_{12} = \frac{\pi^2 c^3}{\hbar\omega^3 t_{sp} n_0^3} u g(\omega) \tag{5.2}$$

The number of atoms undergoing stimulated emissions from E_2 to E_1 per unit volume per unit time will be [see Eqs. (4.16) and (4.18)]

$$\Gamma_{21} = W_{21} N_2 = W_{12} N_2 \tag{5.3}$$

where we have used the fact that the absorption probability is the same as the stimulated emission probability. In addition to the above two transitions, atoms in the level E_2 would also undergo spontaneous transitions from E_2 to E_1. If A_{21} and S_{21} represent the radiative and non-radiative transition[1] rates from E_2 to E_1, then the number of atoms undergoing spontaneous transitions per unit time per unit volume from E_2 to E_1 will be $T_{21} N_2$ where

$$T_{21} = A_{21} + S_{21} \tag{5.4}$$

Thus we may write the rate of change of population of energy levels E_2 and E_1 as

$$\frac{dN_2}{dt} = W_{12}(N_1 - N_2) - T_{21} N_2 \tag{5.5}$$

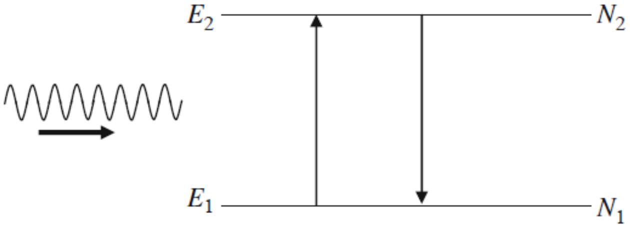

Fig. 5.1 A two-level system

$$\frac{dN_1}{dt} = -W_{12}(N_1 - N_2) + T_{21}N_2 \tag{5.6}$$

As can be seen from Eqs. (5.5) and (5.6)

$$\frac{d}{dt}(N_1 + N_2) = 0$$
$$\Rightarrow N_1 + N_2 = \text{a constant} = N \quad \text{(say)} \tag{5.7}$$

which is nothing but the fact that the total number of atoms N per unit volume is constant. At steady state

$$\frac{dN_1}{dt} = 0 = \frac{dN_2}{dt} \tag{5.8}$$

which gives us

$$\frac{N_2}{N_1} = \frac{W_{12}}{W_{12} + T_{21}} \tag{5.9}$$

Since both W_{12} and T_{21} are positive quantities, Eq. (5.9) shows us that we can never obtain a steady-state population inversion by optical pumping between just two levels.

Let us now have a look at the population difference between the two levels. From Eq. (5.9) we have

$$\frac{N_2 - N_1}{N_2 + N_1} = -\frac{T_{21}}{2W_{12} + T_{21}}$$

or if we write $\Delta N = N_2 - N_1$, we have

$$\frac{\Delta N}{N} = -\frac{1}{1 + 2W_{12}/T_{21}} \tag{5.10}$$