Some continuous distribution

(Uniform, Exponential, Normal)

Uniform distn:

A random variable X is said to have a continuous Uniform distribution on the interval $[a, b]$ where $-\infty < a < b < \infty$ if its pdf can be written as,

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{o.w.} \end{cases}$$

with the cdf. (cumulative distribution function)

$$F(x) = \begin{cases} 0, & x \le a \\ \displaystyle\int_a^x \frac{1}{b-a}\,dx, & a < x < b \\ \displaystyle\int_a^x \frac{1}{b-a}\,dx, & x \geq b \end{cases}$$

$$= \begin{cases} 0, & x \le a \\ \dfrac{x}{b-a}\Big|_a^x, & a < x < b \\ \displaystyle\int_a^b \frac{x}{b-a}\,dx + \int_b^x 0\,dx, & x \geq b \end{cases}$$

$$\therefore \quad F(x) = \begin{cases} 0 , & x \le a \\ \dfrac{x-a}{b-a} , & a < x < b \\ \dfrac{b-a}{b-a} = 1 , & x \geq b \end{cases}$$

Now,

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

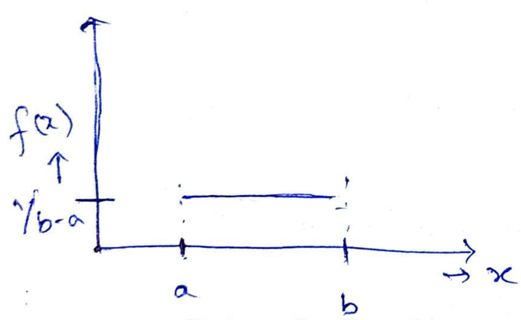
$$= \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{b+a}{2}$$

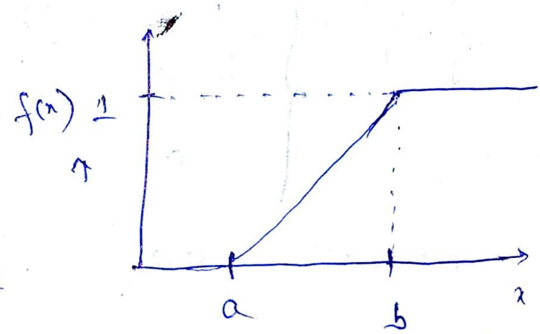
$$V(X) = \frac{(b-a)^2}{12} \quad (\text{Do yourself})$$

by using $V(X) = E(X^2) - \left[E(X)\right]^2$

and $E(X^2) = \displaystyle\int_a^b \frac{x^2}{b-a}\,dx.$



pdf ~~f(x)~~ of Uniform distn



~~F(x)~~ cdf of Uniform distn

Exponential distn:

A random variable X is defined to have an exponential distn with parameter $\lambda > 0$ if its pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty , \lambda > 0 \\ 0 & \text{ow}. \end{cases}$$

The cdf is

$$F(x) = \begin{cases} 0 , & x < 0 \\ \int_0^x \lambda e^{-\lambda x} dx , & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 , & x < 0 \\ \lambda \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^x & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 , & x < 0 \\ 1 - e^{-\lambda x} , & x \geq 0 \end{cases}$$

$$E(X) = \int_0^\infty x \cdot f(x)\, dx = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx .$$

$$= \lambda \int_0^\infty x e^{-\lambda x} dx .$$

$$= 1/\lambda \quad \text{(Do yourself)}$$

[Hints:
$$\int x e^{-\lambda x} dx = x \cdot \int e^{-\lambda x} dx - \int \left[\frac{d}{dx}(x) \cdot \int e^{-\lambda x} dx \right] dx$$
]

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

(Do yourself)

[Hints:
$$\lambda \int x^2 e^{-\lambda x} dx = \lambda \left[x^2 \int e^{-\lambda x} dx - \int \left[\frac{d}{dx}(x^2) \int e^{-\lambda x} dx \right] dx \right]$$
]

Note!

i) r^{th} order raw moment of X is

$$E(X^r) = \mu_r' = \frac{r!}{\lambda^r} \qquad r = 1, 2 \ldots$$

ii) See variance $= \frac{1}{\lambda^2} = \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{\text{Mean}}{\lambda}$

$\therefore$ Variance $>$ Mean if $0 < \lambda < 1$

Variance $=$ mean if $\lambda = 1$

Variance $<$ Mean if $\lambda > 1$

iii) The most important property is that the exponential distⁿ is memoryless.

We can state formally this, as follows.

if $X \sim Exp(\lambda)$,

$$P(X > x + a \mid X > a) = P(X > x)$$

For better understanding, read the following example,

From the point of view of waiting time until arrival of a customer, the memoryless property means that it does not matter how long you ~~wa~~ have waited so far. If you have not observed a customer until time 'a', the distribution of waiting time until the next customer is the same as when you started at time zero.

~~Prove~~ Let us prove the memoryless property for exponential

$$\underbrace{P(X > x + a \mid X > a)}_{\text{called conditional dist}^n.} = \frac{P(X > x+a, X > a)}{P(X > a)}$$

$$= \frac{P(X > x+a)}{P(X > a)}$$

$$= \frac{1 - P(X \le x+a)}{1 - P(X \le a)}$$

$$= \frac{1 - (1 - e^{-\lambda(x+a)})}{1 - (1 - e^{-\lambda a})}$$

$$= e^{-\lambda x}$$

$$= P(X > x)$$

Recall the conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

[For Here $A \Rightarrow X > x+a$

$B \Rightarrow X > a$

and $A \cap B$ means $X > x+a$ and $X > a$ which is obvious $X > x+a$]

iv) Skewness $\gamma_1 = 2$

v) Kurtosis $\gamma_2 = 6$