Poisson distribution:

A random variable X is said to have a Poisson distⁿ with parameter $\lambda\ (>0)$ if its pmf can be written as

$$P(X=x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & x = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

It should be noted that $\sum_{x=0}^{\infty} P(X=x) = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}$

$$= e^{-\lambda}\cdot e^{\lambda} = 1$$

Poisson distⁿ can be applied in the following studies:

i) Number of deaths from a disease such as heart attack or cancer or due to snake bite
ii) Number of printing mistakes at each page of the book.
iii) Number of telephone calls received at a particular telephone exchange in some unit of time to wrong numbers in a telephone exchange.

Moments of the Poisson distribution:

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, e^{-\lambda}\frac{\lambda^x}{x!} = \lambda$$

$$\mu_2' = E(X^2) = E(X(X-1)+X) = E(X(X-1)) + E(X)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, e^{-\lambda}\frac{\lambda^x}{x!} + \lambda$$

$$= e^{-\lambda}\lambda^2\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 . e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu_3' = E(X^3) = E\left(X(X-1)(X-2) + 3X(X-1) + X\right)$$

$$= E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X)$$

$$= \lambda^3 + 3\lambda + \lambda$$

Similarly,

$$\mu_4' = E(X^4) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda \quad \text{(do yourself)}$$

The four central moments are now :

$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \lambda$ (do yourself)

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 3\lambda^2 + \lambda$ (do yourself)

Skewness : $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}$ and $\gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$

kurtosis : $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}$ and $\gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$

i.e. The Poisson distn is always a skewed distn

Recurrence Relation of Poisson distn :

$$\mu_{r+1} = \lambda\left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right)$$

$$\mu_r = E(x-\lambda)^r$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{\lambda^x}{x!}$$

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)e^{-\lambda}\frac{\lambda^x}{x!} + \sum_{x=0}^{\infty}(x-\lambda)^r e^{-\lambda}(-1)\frac{\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty}(x-\lambda)^r e^{-\lambda}\frac{x\lambda^{x-1}}{x!}$$

or $$\frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \sum_{x=0}^{\infty}(x-\lambda)^r e^{-\lambda}\left(\frac{x}{\lambda} - 1\right)\frac{\lambda^x}{x!}$$

or $$\frac{d\mu_r}{d\lambda} + r\mu_{r-1} = \frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1} e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \frac{1}{\lambda}\mu_{r+1}$$

Hence proved.

Mode of Poisson distribution:

$$\frac{p(x)}{p(x-1)} = \frac{\frac{e^{-\lambda}\lambda^{x}}{x!}}{\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x}$$

case 1: when λ is not an integer. Let $\lambda = s+f$ where s is the integer part and f is the fraction.

then $$\frac{p(x)}{p(x-1)} = \frac{s+f}{x} = \begin{cases} >1 & \text{if } x = 0, 1, \dots s \\ <1 & \text{if } x = s+1, s+2 \dots \end{cases}$$

combining these two we get

$p(0) < p(1) < p(2) < \dots < p(s-2) < p(s-1) < p(s) > p(s+1) > p(s+2) \dots$

i.e. the distn is unimodal and integral part of λ is the unique modal value.

case II: when $\lambda = k$ is an integer, then

$p(0) < p(1) < p(2) < \dots < p(k-2) < p(k-1) = p(k) > p(k+1) > p(k+2) \dots$

Thus the Poisson distn is bimodal and two modes are at $(k-1)$ and k.

Note:

i) if X is poisson distn with parameter λ, it can be written as, $X \sim P(\lambda)$

ii) Let X_1 and X_2 be independent RV's, such that $X_1 \sim P(\lambda_1)$ and $X_2 \sim P(\lambda_2)$ then ~~$X_1 + X_2$~~

$X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$

iii) If X_i $(i = 1, 2 \dots k)$ are independent Poisson variates with parameters λ_i $(i = 1, 2 \dots k)$ then their sum

$$\sum_{i=1}^{k} X_i \sim P\left(\sum_{i=1}^{k} \lambda_i\right)$$

(4) Poisson distⁿ is a limiting case of the Binomial distⁿ under the following conditions:

i) n, the number of trials is indefinitely large, i.e. $n \to \infty$

ii) p, the constant probability of success for each trial is indefinitely small, i.e. $p \to 0$

iii) $np = \lambda$ is finite.

Thus $p = \frac{\lambda}{n}$

The prob. of x successes in a series of n independent trials is

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \quad ; x = 0, 1, 2 \ldots n$$

Now,

$$\lim_{n\to\infty} b(x; n, p) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

Using Stirling's approximation for $n!$ as $n \to \infty$

$$\lim_{n\to\infty} n! \simeq \sqrt{2\pi}\, e^{-n}\, n^{n+1/2}$$

we get,

$$\lim_{n\to\infty} b(x; n, p) = \lim_{n\to\infty} \left\{ \frac{\sqrt{2\pi}\, e^{-n}\, n^{n+1/2}}{\sqrt{2\pi}\, e^{-x}\, x^{x+1/2}} \right.$$

$$= \lim_{n\to\infty} \left\{ \frac{\sqrt{2\pi}\, e^{-n}\, n^{n+1/2}}{x!\,\sqrt{2\pi}\, e^{-(n-x)}\, (n-x)^{n-x+1/2}} \right\} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^x x!} \lim_{n\to\infty} \left\{ \frac{n^{n-x+1/2}}{(n-x)^{n-x+1/2}} \right\} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^x x!} \lim_{n\to\infty} \frac{\left(1-\frac{\lambda}{n}\right)^{n-x}}{\left(1-\frac{x}{n}\right)^{n-x+1/2}}$$

$$= \frac{\lambda^x}{e^x\, x!} \quad \frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}{\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n \cdot \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-x+1/2}}$$

As we know, $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ and $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{\alpha} = 1$

$$\lim_{n\to\infty} p(x; n, p) = \frac{\lambda^x}{e^x\, x!} \cdot \frac{e^{-\lambda} \cdot 1}{e^{-x} \cdot 1} = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2, \ldots \infty$$