Transformation of Co-ordinates

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What is Geometry ?

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➢ **Euclid** (300 BC) was an ancient Greek mathematician in Alexandria (familiar with Euclid of Alexandria), Egypt. Due to his groundwork in math, he is often referred to as the **'Father of Geometry'. Euclid's** most well-known collection of works, called **Elements,** outlines some of the most fundamental principles of geometry.

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☞ Method of study may be (i) **algebraic**, (ii) **non-algebraic**. In algebraic method at first we have to establish relationship between the elements of geometry and the elements of algebra.

Transformation of axes

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(iii) Combination of Translation and Rotation **(Rigid motion.)**

Translation

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Let (x, y) be the co-ordinates of P w.r.t. rectangular axes OX and OY and (x', y') be the co-ordinates of it w.r.t. a new set of axes $O'X'$ and $O'Y'$ which are parallel to the original axes OX and OY respectively.

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Let (α, β) be the coordinates of the new origin O' w.r.t. axes OX and OY. PM is perpendicular to OX and it meets $O'X'$ at N. $O'L$ is perpendicular to OX.

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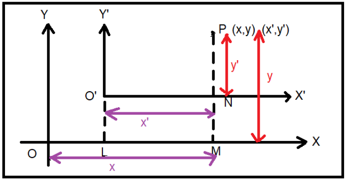

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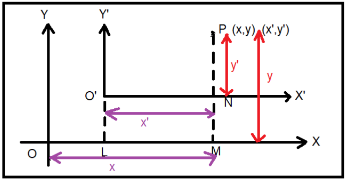

$\therefore \ OM = x, \quad PM = y$
$\quad O'N = x', \quad PN = y'$
$\quad OL = \alpha, \quad O'L = \beta$

Now
$x = OM = OL + LM = OL + O'N = \alpha + x'$,
$y = PM = PN + NM = PN + O'L = y' + \beta$
Hence, the required transformation formulae are given by
$\mathbf{x} = \mathbf{x}' + \alpha, \ \mathbf{y} = \mathbf{y}' + \beta$
This transformation is also known as **translation** or **parallel displacement.**

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In the equation of a locus referred to original system of axes (x, y) will be replaced by $(x' + \alpha, y' + \beta)$ when the equation is referred to new pair of axes.
Inversely, (x', y') will be replaced by $(x - \alpha, y - \beta)$.

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In matrix notation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

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As $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an orthogonal matrix, therefore **translation** is an **orthogonal translation.**

Rotation

Rotation

Let the original axes OX and OY be rotated through an angle θ in the anti-clockwise direction. In the adjoining figure OX' and OY' are the new set of axes. Let (x, y) and (x', y') be the co-ordinates of the same point P referred to OX, OY and OX', OY' respectively.

PN and PM are perpendicular to OX and OX' respectively. Here $\angle X'OX = \theta$.

From the figure, $ON = x,\ NP = y,\ OM = x',\ MP = y'$.

$\mathbf{x} = ON = OK - NK = OK - TM = \mathbf{x}' \cos\theta - \mathbf{y}' \sin\theta$

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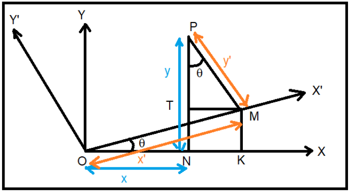

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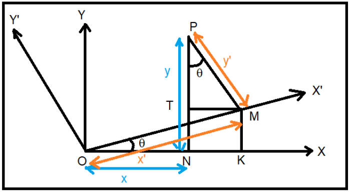

Hence the change from (x, y) to (x', y') is given by $\begin{cases} x = x'\cos\theta - y'\sin\theta, \\ y = x'\sin\theta + y'\cos\theta \end{cases}$
From above equations, we can easily deduce that $\begin{cases} x' = x\cos\theta + y\sin\theta, \\ y' = y\cos\theta - x\sin\theta \end{cases}$
This transformation is known as **rotation.**

Rotation

In matrix notation, we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

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Both of the tansformations can be remembered by the scheme

	x′	y′
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Rigid Motion

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Let the co-ordinates (x, y) changed to (x', y') by translation and (x', y') to (x'', y'') by transformation of rotation.

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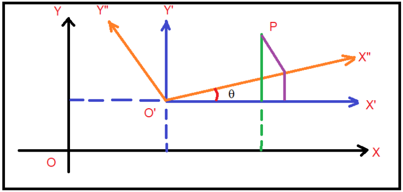

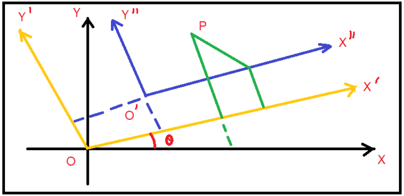

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The general orthogonal transformation formulae :

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Motivation

1. To remove the linear terms, i.e., the terms containing x or y or both, we apply translation.

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If $\alpha = 0 = \beta$, then it is rotation.

Motivation

1. To remove the linear terms, i.e., the terms containing x or y or both, we apply translation.

2. to remove the quadratic terms, i.e., the terms containing x^2 or y^2 or xy, we apply rotation.

Examples

Ex. 1 : Find the transformation which transforms the equation $x^2 + y^2 - 2x + 14y + 20 = 0$ into $x'^2 + y'^2 - 30 = 0$.

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Let us apply the translation $x = x' + \alpha, \quad y = y' + \beta$.
$\therefore$ The equation $x^2 + y^2 - 2x + 14y + 20 = 0$ transformed' to
$(x' + \alpha)^2 + (y' + \beta)^2 - 2(x' + \alpha) + 14(y' + \beta) + 20 = 0$
$x'^2 + y'^2 + 2(\alpha - 1)x' + 2(\beta + 7)y' + \alpha^2 + \beta^2 - 2\alpha + 14\beta + 20 = 0$
Choose $\alpha, \ \beta$ so that $\alpha - 1 = 0$ and $\beta + 7 = 0$
$\therefore \quad \alpha = 1, \quad \beta = -7$
$\therefore$ The transformed equation becomes $x'^2 + y'^2 - 30 = 0$.
Hence the transformation is the translation $x = x' + 1, \quad y = y' - 7$.

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Ex. 2 : Find the rotation so that the equation $3x^2 - 3y^2 + 10xy = 0$ will reduce to the form $ax^2 + by^2 = 0$.

Ans. : Here the sum of the coefficients x^2 and y^2 is zero.

Examples

$\therefore$ Required angle of rotation is $\frac{\pi}{4}$. $[\because \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}]$
The transformation of rotation is given by
$x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'; \quad y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$
$\therefore$ The given equation transformed to
$3(\frac{x'-y'}{\sqrt{2}})^2 + 10\frac{(x'-y')(x'+y')}{2} - 3(\frac{x'+y'}{\sqrt{2}})^2 = 0$
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Ex. 3 : Applying suitable rotation of coordinate axes transform the equation $8x^2 - 12xy + 17y^2 = 0$ to an equation containing no term of xy.

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Ans. : The required angle of rotation θ is given by
$\tan 2\theta = \frac{-12}{8-17} = \frac{4}{3}$ [Formula : $\tan 2\theta = \frac{2h}{a-b}$]
Taking θ as an accute angle, we get, $\tan\theta = \frac{1}{2}$.
$\therefore \sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}$
$\therefore$ The suitable rotation is given by $x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y'$, $y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'$.
$\therefore$ In the new coordinate system the given equation reduces to
$8(\frac{2x'-y'}{\sqrt{5}})^2 - 12(\frac{2x'-y'}{\sqrt{5}})(\frac{x'+2y'}{\sqrt{5}}) + 17(\frac{x'+2y'}{\sqrt{5}})^2 = 0$
or, $x'^2 + 4y'^2 = 0$.

Examples

Ex. 4 : Find the angle through which the axes must be turned so that the equation $lx + my + n = 0$ $(m \neq 0)$ may reduce to the form $by + c = 0$.

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Ans. : Let the axes be turned through an angle θ.
$\therefore$ the transformation of rotation is $x = x' \cos\theta - y' \sin\theta, \; y = x' \sin\theta + y' \cos\theta$
$\therefore$ $lx + my + n = 0$ transformed to
$l(x' \cos\theta - y' \sin\theta) + m(x' \sin\theta + y' \cos\theta) + n = 0$
or, $(l\cos\theta + m\sin\theta)x' + (m\cos\theta - l\sin\theta)y' + n = 0$
It will reduce to the form $by + c = 0$
$\therefore$ $l\cos\theta + m\sin\theta = 0$ or, $\tan\theta = -\frac{l}{m}$ or, $\theta = \tan^{-1}(-\frac{l}{m})$.

Ex. 5 : P(9,-1) and Q(-3,4) are two points. If the origin be shifted to the point P and the coordinate axes be rotated so that the positive direction of the new x-axis agrees with the direction of the segment $\overline{PQ}$ then find the coordinate transformation formula.

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Ans. : For translation and rotation
$x = x' \cos\theta - y' \sin\theta + \alpha,\ y = x' \sin\theta + y' \cos\theta + \beta$

By the given condition, $\alpha = 9,\ \beta = -1$.

Examples

In the new system the other point lies on the x-axis. Therefore, the coordinates of this point are of the form $(c, 0)$. Thus putting $x' = c,\ y' = 0$, we have,
$-3 = c\cos\theta + 9$ and $4 = c\sin\theta - 1$
or, $c\cos\theta = -12$ and $c\sin\theta = 5$
or, $\tan\theta = -\frac{5}{12}$ i.e., $\theta = \tan^{-1}(-\frac{5}{12})$.
$\therefore\ \frac{\sin\theta}{5} = \frac{\cos\theta}{-12} = \frac{1}{13}$ or, $\sin\theta = \frac{5}{13},\ \cos\theta = -\frac{12}{13}$
Thus the required formulae are
$x = -\frac{12}{13}x' - \frac{5}{13}y' + 9,\ y = \frac{5}{13}x' - \frac{12}{13}y' - 1.$

Ex. 6 : When the axes are turned through an angle, the expression $(ax + by)$ becomes $(a'x' + b'y')$ referred to new axes; Show that $a^2 + b^2 = a'^2 + b'^2$.

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Ans. : Let, the axes be turned through an angle θ.
$\therefore$ the transformation of rotation is
$x = x'\cos\theta - y'\sin\theta, y = x'\sin\theta + y'\cos\theta$
$\therefore\ ax + by = a(x'\cos\theta - y'\sin\theta) + b(x'\sin\theta + y'\cos\theta)$
$= (a\cos\theta + b\sin\theta)x' + (b\cos\theta - a\sin\theta)y' = a'x' + b'y'$
$\therefore\ a' = a\cos\theta + b\sin\theta$ and $b' = b\cos\theta - a\sin\theta$
$\therefore\ a'^2 + b'^2 = (a\cos\theta + b\sin\theta)^2 + (b\cos\theta - a\sin\theta)^2 = a^2 + b^2.$

Examples

Ex. 7 : Show that there is one point whose co-ordinates do not alter due to a rigid motion.

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Ans. : Let the transformation of rigid motion be
$x = px' - qy' + r, \quad y = qx' + py' + s,$ where $p^2 + q^2 = 1,$ and $p \neq 1, \ (r, s) \neq (0, 0)$
For unalteration of coordinates, we have, $x = px - qy + r, \quad y = qx + py + s$
or, $(1 - p)x + qy = r, \quad -qx + (1 - p)y = s$
This is a system of non-homogeneous equations as $(r, s) \neq (0, 0)$
Now, det(coefficient matrix)= $\begin{vmatrix} 1-p & q \\ -q & 1-p \end{vmatrix} = 2(1 - p) \neq 0.$
$\therefore$ there is always one fixed point under rigid motion. Therefore the point is

$$\frac{x}{\begin{vmatrix} r & q \\ s & 1-p \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1-p & r \\ -q & s \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1-p & q \\ -q & 1-p \end{vmatrix}}$$

$\therefore \ x = \frac{(1-p)r-qs}{2(1-p)}, \ y = \frac{(1-p)s+rq}{2(1-p)}.$

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If the expression $ax^2 + 2hxy + by^2$ transformed to $a'x^2 + 2h'xy + b'y^2$ by the transformation of rotation then
(i) $a' + b' = a + b$
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Proof : Let the transformation of rotation be
$x = x'\cos\theta - y'\sin\theta,\ y = x'\sin\theta + y'\cos\theta$
$\therefore\ ax^2 + 2hxy + by^2$ transformed to $a'x'^2 + 2h'x'y' + b'y'^2$, where
$a' = a\cos^2\theta + h\sin 2\theta + b\sin^2\theta$
$h' = \frac{1}{2}(b - a)\sin 2\theta + h\cos 2\theta$
$b' = a\sin^2\theta - h\sin 2\theta + b\cos^2\theta$
$\therefore\ a' + b' = a(\cos^2\theta + \sin^2\theta) + b(\sin^2\theta + \cos^2\theta) = a + b.$

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$\therefore\ a' + b' = a(\cos^2\theta + \sin^2\theta) + b(\sin^2\theta + \cos^2\theta) = a + b.$
Again $4(a'b' - h'^2) = \{a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)\}\{a(1-\cos 2\theta) - 2h\sin 2\theta + b(1+\cos 2\theta)\} - \{(b-a)\sin 2\theta + 2h\cos 2\theta\}^2$
$= (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2 - \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2$
$= (a+b)^2 - \{(a-b)^2 + 4h^2\} = 4(ab-h)^2$

Invariants

Alternative Proof :

Consider the rotation $x = px' - qy',\ y = qx' + py',$ where $p^2 + q^2 = 1$.

$\therefore$ the expression $ax^2 + 2hxy + by^2$ transformed to

$a(px' - qy')^2 + 2h(px' - qy')(qx' + py') + b(qx' + py')^2$

i.e., $(ap^2 + 2hpq + bq^2)x'^2 + 2\{-apq + h(p^2 - q^2) + bpq\}x'y' + (aq^2 - 2hpq + bp^2)y'^2$

$\therefore$ by the given information

$$\begin{aligned} a' &= ap^2 + 2hpq + bq^2 \\ h' &= (b-a)pq + h(p^2 - q^2) \\ b' &= aq^2 - 2hpq + bp^2 \end{aligned}$$

$\therefore\ a' + b' = a + b \qquad \because p^2 + q^2 = 1$

and

$ab - h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} p & q \\ -q & p \end{vmatrix} \begin{vmatrix} a & h \\ h & b \end{vmatrix} \begin{vmatrix} p & q \\ -q & p \end{vmatrix}$ $[\because \begin{vmatrix} p & q \\ -q & p \end{vmatrix} = p^2 + q^2 = 1]$

$$= \begin{vmatrix} ap + hq & hp + bq \\ -aq + hp & -hq + bp \end{vmatrix} \begin{vmatrix} p & q \\ -q & p \end{vmatrix}$$

$$= \begin{vmatrix} ap^2 + 2hpq + bq^2 & h(p^2 - q^2) + (b-a)pq \\ h(p^2 - q^2) + (b-a)pq & aq^2 - 2hpq + bp^2 \end{vmatrix}$$

$$= \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix} = a'b' - h'^2.$$

Examples

Ex. 8 : If $A_i(x_i, y_i),\ i = 1, 2, 3$ be three points in a plane, then show that $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ remains invariant under a rigid motion.

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Ex. 8 : If $A_i(x_i, y_i)$, $i = 1, 2, 3$ be three points in a plane, then show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ remains invariant under a rigid motion.

Or, Under any rigid motion the area of a triangle is an invariant.

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Or, Under any rigid motion the area of a triangle is an invariant.

Ans. : Let the transformation of rigid motion be

$x = px' - qy' + r, \quad y = qx' + py' + s$, where $p^2 + q^2 = 1$.

Now $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} px_1' - qy_1' + r & qx_1' + py_1' + s & 1 \\ px_2' - qy_2' + r & qx_1' + py_1' + s & 1 \\ px_3' - qy_3' + r & qx_1' + py_1' + s & 1 \end{vmatrix}$

$= p \begin{vmatrix} x_1' & qx_1' + py_1' + s & 1 \\ x_2' & qx_1' + py_1' + s & 1 \\ x_3' & qx_1' + py_1' + s & 1 \end{vmatrix} - q \begin{vmatrix} y_1' & qx_1' + py_1' + s & 1 \\ y_2' & qx_1' + py_1' + s & 1 \\ y_3' & qx_1' + py_1' + s & 1 \end{vmatrix} +$

$r \begin{vmatrix} 1 & qx_1' + py_1' + s & 1 \\ 1 & qx_1' + py_1' + s & 1 \\ 1 & qx_1' + py_1' + s & 1 \end{vmatrix}$

Examples

$$= p^2 \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} - q^2 \begin{vmatrix} y_1' & x_1' & 1 \\ y_2' & x_2' & 1 \\ y_3' & x_3' & 1 \end{vmatrix}$$

$$= (P^2 + q^2) \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}$$

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6.


Thank you