ELECTROMAGNETIC THEORY

Genaralisation of Ampere's Law

Write down Maxwell's electromagnetic field equations and explain physical significance of each · [V.U - 2006, 2007, 2009]

There are four fundamental equations of electromagnetic theory, Known as Maxwell's electromagnetic field equations. They are —

1. $\vec{\nabla}\cdot\vec{D}=\rho$ ⇛ Differential form of Gauss's law in electrostatics.
2. $\vec{\nabla}\cdot\vec{B}=0$ ⇛ Differential form of Gauss's law in magnetostatics.
3. $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ ⇛ Differential form of Faraday's law in E.M. induction.
4. $\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$ ⇛ Differential form of modified Ampere's law.

Where, $\vec{D}$ = electric displacement vector [unit – Coulomb/m^2]
ρ ⇒ charge density ...... [unit – Coulomb/m^3]
$\vec{B}$ ⇒ magnetic induction ..... [unit – Weber/m^2]
$\vec{E}$ ⇒ electric field intensity ... [unit – Volt/m]
$\vec{H}$ ⇒ magnetic field intensity .. [unit – Ampere/m-turn]

Derivation of Maxwell's equations :

i) Maxwell's 1st equation : $\vec{\nabla}\cdot\vec{D}=\rho$:

Let us consider a surface s bounding a volume v in a dielectric medium. According to Gauss's law in electrostatics,

$$\oiint \vec{D}\cdot\hat{n}\,ds = q$$

If ρ be the free charge density $q=\iiint \rho\, dv$.

$\therefore \oiint \vec{D}\cdot\hat{n}\, ds = \iiint \rho\, dv$.

$\Rightarrow \iiint \vec{\nabla}\cdot\vec{D}\, dv = \iiint \rho\, dv$ according to Gauss's divergence theorem.

$\Rightarrow \iiint (\vec{\nabla}\cdot\vec{D} - \rho)\, dv = 0$

Since volume taken is arbitrary, $dv \neq 0$

$\therefore \vec{\nabla}\cdot\vec{D} - \rho = 0$

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho}$

ii) Maxwell's 2nd equation ; $\vec{\nabla}\cdot\vec{B} = 0$

Since isolated magnetic poles do not exist; hence magnetic current due to them have no physical significance. Thus the number of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it.

$\therefore \oiint \vec{B}\cdot\hat{n}\, ds = 0$.

$\Rightarrow \iiint \vec{\nabla}\cdot\vec{B}\, dv = 0$ according to Gauss's div. theorem.

Since $dv \neq 0$ $\therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

iii) Maxwell's 3rd equation ; $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

According to Faraday's law of electromagnetic induction, the induced emf is the negative rate of change of magnetic flux i.e $e = -\frac{\partial\phi}{\partial t}$

$$= -\frac{\partial}{\partial t}\iint \vec{B}\cdot\hat{n}\, ds$$

$$= -\iint \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\, ds \quad \text{——} ①$$

Again emf can be computed by calculating workdone to bring a unit +ve charge round a closed path once.

i.e $e = \oint \vec{E}\cdot\vec{dl}$ —② as $\vec{F} = q\vec{E} = \vec{E}$.

From equation (1) and (2),

$$\oint \vec{E}\cdot\vec{dl} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\,ds$$

$$\Rightarrow \iint \vec{\nabla}\times\vec{E}\cdot\hat{n}\,ds = -\iint \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\,ds$$ according to Stoke's Theorem

$$\Rightarrow \iint \left(\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot\hat{n}\,ds = 0$$

Since $ds \neq 0$, $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

iv) <u>Maxwell's 4th equation;</u> $\underline{\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$ [<u>V.U-1994</u>].

According to Ampere's circuital law,

$$\oint \vec{H}\cdot\vec{dl} = i$$

If $\vec{J}$ be the conducting current density then $i = \oiint \vec{J}\cdot\hat{n}\,ds$.

$$\therefore \oint \vec{H}\cdot\vec{dl} = \oiint \vec{J}\cdot\hat{n}\,ds$$

$$\Rightarrow \oiint \vec{\nabla}\times\vec{H}\cdot\hat{n}\,ds = \oiint \vec{J}\cdot\hat{n}\,ds$$ according to Stoke's law.

$$\Rightarrow \oiint \left(\vec{\nabla}\times\vec{H} - \vec{J}\right)\cdot\hat{n}\,ds = 0$$

Since $ds \neq 0$, $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}}$

Taking divergence on both sides, we get.

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = \vec{\nabla}\cdot\vec{J}$$

$\Rightarrow \underline{\vec{\nabla}\cdot\vec{J} = 0}$ —— (2) as $\vec{\nabla}\cdot\vec{\nabla}\times$ any vector $= 0$.

Again according to equation of continuity,

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$\Rightarrow \underline{\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}}$ —— (3)

If $\frac{\partial \rho}{\partial t} = 0$, Ampere's circuital law agree's with equation of continuity. But <u>for time varying field</u> $\left(\frac{\partial \rho}{\partial t} \neq 0\right)$, Ampere's circuital law does not agree with equation of continuity. Hence modification was made by Maxwell for time varying field in Ampere's law.

Maxwell's modification of Ampere's circuital law —

Maxwell was introduced an additional term $\vec{J}_d$ in Ampere law to include the time varying field,

i.e $\vec{\nabla}\times\vec{H} = \vec{J} + \vec{J}_d$

Taking divergence on both sides,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = \vec{\nabla}\cdot(\vec{J}+\vec{J}_d)$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = -\vec{\nabla}\cdot\vec{J} \quad \text{as } \vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = +\frac{\partial\rho}{\partial t} \quad [\because \text{ Equation of continuity } \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}]$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \frac{\partial}{\partial t}[\vec{\nabla}\cdot\vec{D}] \quad (\text{Maxwell's 1st equation } \vec{\nabla}\cdot\vec{D} = \rho)$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \vec{J}_d = \frac{\partial\vec{D}}{\partial t}$$

Hence modified Ampere's law become,

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}}$$

Interpretation:

* The additional term $\frac{\partial\vec{D}}{\partial t}$ is known as displacement current density as it is time rate of change of electric displacement vector. and adds with current density.

What is displacement current ?

Displacement current: Displacement current is the current which is set-up in a dielectric mediur ($\sigma=0$) due to variation of induced displacement charge produced by the changing electric field applied across the dielectric.

Characteristics of displacement current :

i) Displacement current is the current only in the sense that it produces a magnetic field. It ~~is~~ has none of other properties of current, since it is not linked with motion of charge.

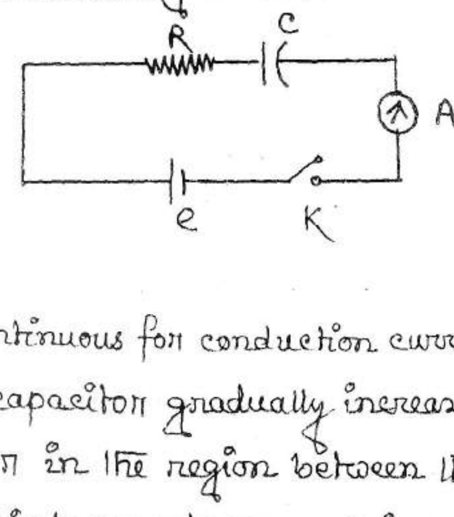
ii) The magnitude of displacement current density is the rate of change of ~~change~~ electric displacement vector i.e

$$\vec{J_d} = \frac{\partial \vec{D}}{\partial t}.$$

iii) Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current.

iv) In a good conductor $\vec{J_d}$ is negligible compared to $\vec{J}$.

Realize the existance of displacement current by an experiment :



A capacitor (c) is charged through a resistance (R) by a battery of emf e. In this case there is a discontinuity between the plates of the capacitor. Thus the path is not continuous for conduction current. As the time passes the charge on the capacitor gradually increases. Hence the electric displacement vector in the region between the plates increases. As a result the displacement current increases and makes the path continuous.

Starting with the differential forms of Faraday's law in electromagnetic induction and Ampere's modified circuital law obtain the differential forms of Gauss's law of magnetostatics and electrostatics respectively. V.U-1994

Solution: i) Differential form of Faraday's law in electromagnetic induction, $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

Taking divergence on both sides,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{E} = -\vec{\nabla}\cdot\frac{\partial \vec{B}}{\partial t}.$$

$\Rightarrow \vec{\nabla}\cdot\frac{\partial \vec{B}}{\partial t} = 0$ as div. curl of any vector is zero.

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0.$$

Thus either $\vec{\nabla}\cdot\vec{B} = 0$ or constant.

An isolated magnetic pole do not exist in nature, thus $\vec{\nabla}\cdot\vec{B} \neq 0$

Thus $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ i.e Gauss's law in magnetostatics.

ii) Modified Ampere's circuital law,

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

Taking divergence on both sides,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}.$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0$$

~~From Ohm's law,~~ $\cancel{\vec{J} = \sigma\vec{E}}$

~~From Maxwell's 1st equation,~~ $\cancel{\vec{\nabla}\cdot\vec{D} = \rho}$

$\therefore \cancel{\vec{\nabla}\cdot\sigma\vec{E} + \frac{\partial \rho}{\partial t} = 0}$ $\qquad \cancel{\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon}$

$\Rightarrow \cancel{\sigma\cdot\rho/\epsilon + \frac{\partial \rho}{\partial t} = 0}$.

$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = -\vec{\nabla}\cdot\vec{J} = \frac{\partial \rho}{\partial t}$ (using equation of continui

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho}$ i.e Gauss's law in electrostatics.

PROBLEM

Problem (1) V.U-2006 : Starting from Maxwell's equation for a homogeneous dielectric medium of conductivity σ derive an expression for the growth or decay of electric charge density and hence define relaxation time. V.U-2006

Solution: Maxwell's equation, $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Taking divergence on both sides,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0 \quad \text{—— (1)}$$

From Ohm's law $\vec{J} = \sigma\vec{E}$

From Maxwell's 1st equation $\vec{\nabla}\cdot\vec{D} = \rho$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon$$

$$\therefore \vec{\nabla}\cdot\sigma\vec{E} + \frac{\partial\rho}{\partial t} = 0$$

$$\Rightarrow \sigma\,\rho/\epsilon + \frac{\partial\rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = -\frac{\sigma}{\epsilon}\rho$$

$$\Rightarrow \int_{\rho_0}^{\rho}\frac{\partial\rho}{\rho} = -\frac{\sigma}{\epsilon}\int_0^t \partial t$$

$$\Rightarrow \ln \rho/\rho_0 = -\frac{\sigma t}{\epsilon}$$

$$\Rightarrow \boxed{\rho = \rho_0 e^{-\sigma t/\epsilon}}$$

Thus charge density decays exponentially with time.

Relaxation time :

At $t = \tau$ (relaxation time), $\rho = \rho_0/e$

Then $\rho_0/e = \rho_0 e^{-\sigma/\epsilon\,\tau}$

$$\Rightarrow \frac{\sigma}{\epsilon}\tau = 1$$

$$\Rightarrow \boxed{\tau = \frac{\epsilon}{\sigma}}$$

Problem (2): Show That equation of continuity $\vec{\nabla}.\vec{J}+\frac{\partial \rho}{\partial t}=0$ is contained in Maxwell's equation.
V.U-2001

V.U - 1995, 2001

Solution: From Maxwell's 4th equation,

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial \vec{D}}{\partial t}$$

Taking divergence on both sides,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{H}=\vec{\nabla}\cdot\vec{J}+\vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})=0 \quad \text{as } \vec{\nabla}\cdot\vec{\nabla}\times\vec{H}=0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J}+\frac{\partial \rho}{\partial t}=0} \quad \text{as } \vec{\nabla}\cdot\vec{D}=\rho$$

We get equation of continuity.

Problem (3): Compare the magnitudes of conduction and displacement current densities in a good conductor for which $\sigma = 10^7$ mho/m., $\epsilon_r = 2$, electric field intensity is a function of $\sin 120\pi t$. Comment on the result obtained.
V.U-1995

Solution: Conduction current density $J_c = \sigma E = \sigma E_0 \sin 120\pi t$.

Displacement current density

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E)$$

$$= \epsilon \frac{\partial E}{\partial t}$$

$$= \epsilon E_0\, 120\pi \times \cos 120\pi t$$

Ratio of magnitudes,

$$\frac{|J_c|}{|J_D|} = \frac{\sigma E_0}{120\pi\,\epsilon E_0} = \frac{10^7}{120\times 3.14\times 8.85\times 10^{-12}\times 2} \quad \text{as } \epsilon = \epsilon$$

$$= 1.4987\times 10^5 = 149870 \times 10^{10}$$

Comment: Conduction current is much greater than displacem current in a good conductor.

Problem (4) : V.U - 1998, V.U - 2001

A parallel plate capacitor having area of each plate $10\ m^2$, plate separation 0.04 mm and $\epsilon_r = 50$ is charged to a potential of $200 \cos \pi t$ volts. Find the displacement current.

Solution : Given $A = 10\ m^2$

$d = 0.04\ mm = 4 \times 10^{-5}\ m$.

$\epsilon_r = 50$

$V = 200 \cos \pi t$ volts.

Displacement current
$$I_d = A \times J_d$$
$$= A \times \frac{\partial D}{\partial t}$$
$$= A \times \frac{\partial}{\partial t}(\epsilon E).$$
$$= \epsilon A \frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$
$$= \frac{\epsilon_0 \epsilon_r A}{d} \cdot \frac{\partial V}{\partial t}.$$
$$= \frac{8.85 \times 10^{-12} \times 50 \times 10}{4 \times 10^{-5}} \times 200 \times \pi \times \sin \pi t.$$
$$= \underline{3.4769 \times 10^{-2} \times \sin \pi t \text{ amp}}$$

Problem (5) : V.U - 2003

Maximum displacement current in a medium of dielectric constant 10 is equal to maximum conduction current at a frequency of 10 MHz. What is the conductivity of the medium?

Solution : $\epsilon_r = 10$

$\nu = 10\ MHz = 10^7\ Hz$.

Conduction current $J_c = \sigma E = \sigma E_0 \sin \omega t$.

$$\therefore (J_c)_{max} = \sigma E_0 \quad \text{——— (1)}$$

Displacement current $J_d = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t} = \epsilon \omega E_0 \cos \omega t$.

$$\therefore (J_d)_{max} = \epsilon_0 \epsilon_r \omega E_0 \quad \text{——— (2)}$$

When, maximum displacement current = maximum conduc current

i.e $\sigma E_0 = \epsilon_0 \epsilon_r \omega E_0$

$$\Rightarrow \sigma = \epsilon_0 \epsilon_r \times 2\pi \gamma$$
$$= 8.85 \times 10^{-12} \times 10 \times 2 \times 3.14 \times 10^{7}$$
$$= 88.5 \times 6.28 \times 10^{-5}$$
$$= \ldots\ldots \text{ mho/m}.$$

Problem (6): Find the frequency of conduction current for which maximum conduction current density is equal to maximum displacement current density.

Solution:

$$\left.\begin{array}{l}(J_c)_{max} = \sigma E_0 \\ (J_d)_{max} = \epsilon \omega E_0\end{array}\right\} \text{ From problem (5)}$$

When $(J_c)_{max} = (J_d)_{max}$

$$\Rightarrow \sigma E_0 = \epsilon \omega E_0$$
$$\Rightarrow \sigma = \epsilon \times 2\pi \gamma$$
$$\Rightarrow \boxed{\gamma = \frac{\sigma}{2\pi\epsilon}} \text{ c.p.s.}$$

Problem (7): A long cylindrical conductor of radius R a
V.U-1999
V.U-2006

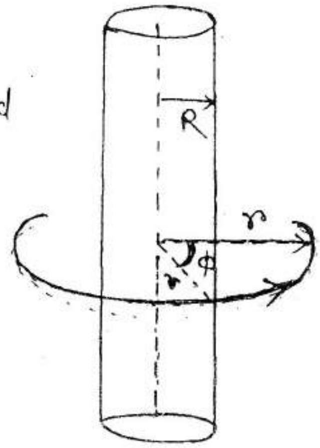
$\sigma = \infty$ carries a current $I = I_0 \sin \omega t$. As a fu of radius r (for $r<R$ and $r>R$) find –

i) The conduction current density $J_c(r)$

ii) The displacement current density $J_d(r)$

iii) The magnetic flux density $B(r)$

Solution : In a good conductor, the distribution of charge is quickly dispersed to the surface. Hence we may regard that, the interior of the conductor to be uncharged i.e. no permanent charge density.



Thus $\vec{\nabla}\cdot\vec{E} = 0$ as $\rho = 0$.

$$\Rightarrow E(r) = \text{constant}.$$

i) ∴ Displacement current density,

$$J_d(r) = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = 0 \text{ for all values of } r.$$

ii). Conduction current density,

$$J_c(r) = \frac{I}{\pi R^2} = \frac{I_0 \sin\omega t}{\pi R^2} \text{ for } r \leq R$$

$$= 0 \qquad \text{for } r \geq R$$

iii) From Ampere's circuital law,

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Taking surface integral, $\iint \vec{\nabla}\times\vec{B}\cdot\hat{n}\, ds = \iint \mu_0 \vec{J}\cdot\hat{n}\, ds$.

According to Stoke's theorem, $\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint \vec{J}\cdot\hat{n}\, ds$.

In cylindrical case, by virtue of symmetry, the field B will have only ϕ component,

$$\oint_0^{2\pi} B_\phi\, r\, d\phi = \mu_0 \int_{r=0}^{r}\int_{\phi=0}^{2\pi} J\, r\, d\phi\cdot dr \text{ for } r<R.$$

$$\Rightarrow B_\phi\, r \times 2\pi = \mu_0 J \times 2\pi \times \frac{r^2}{2}$$

$$\Rightarrow B_\phi = \frac{\mu_0 I}{\pi R^2}\times\frac{r}{2} = \frac{\mu_0 I r}{2\pi R^2}$$

$$\therefore \boxed{B_\phi = \frac{\mu_0 I}{2\pi R^2} r} \text{ for } r<R$$

For $r>R$, total current enclosed is I. ∴ $\int_0^{2\pi} B_\phi\, r\, d\phi = \mu_0 I$.

$$\therefore \boxed{B_\phi = \frac{\mu_0 I}{2\pi r}}$$

State and establish Poynting theorem : V.U-2006, 2007, 2008

Poynting theorem : Poynting theorem states that the time rate of change electromagnetic energy within a certain volume plus the time rate of change of energy flowing out through the boundary surface is equal to the power transferred into the electromagnetic field.

Mathematically, $\frac{\partial}{\partial t}\iiint u\,dv + \oiint \vec{P}\cdot\hat{n}\,ds = -\iiint \vec{E}\cdot\vec{J}\,dv$

Where $\vec{P}$ is the Poynting vector.
u is the electromagnetic energy density.

Establish : Electromagnetic energy density (u)
= electrostatic energy density (u_e)
+ magnetostatic energy density (u_m)

$$u = \frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{H} \quad \text{—— (1)}$$

Also the Maxwell's electromagnetic equations,

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{—— (2)}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{—— (3)}$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{—— (4)}$$
$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \text{—— (5)}$$

Taking scalar product of eqn. (4) with $\vec{H}$ and equation (5) with $\vec{E}$, we get ——

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \quad \text{—— (6)}$$
$$\vec{E}\cdot\vec{\nabla}\times\vec{H} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \quad \text{—— (7)}$$

Subtracting equation ⑦ from equation ⑥, we get –

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) = -\vec{E}\cdot\vec{J} \quad \text{——} ⑧$$

Again $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})\right]$

$$= \frac{1}{2}\left[\frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \vec{D}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} + \vec{B}\cdot\frac{\partial\vec{H}}{\partial t}\right]$$

$$= \frac{1}{2}\left[\frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \epsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} + \mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}\right]$$

$$= \frac{1}{2}\left[\frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]$$

$$= \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\therefore \vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{\partial u}{\partial t} = -\vec{E}\cdot\vec{J}$$

Integrating over the volume,

$$\iiint \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv + \iiint\frac{\partial u}{\partial t}\,dv = -\iiint\vec{E}\cdot\vec{J}\,dv$$

According to Gauss's divergence theorem (on 1st term),

$$\frac{\partial}{\partial t}\iiint u\,dv + \oiint(\vec{E}\times\vec{H})\cdot\hat{n}\,ds = -\iiint\vec{E}\cdot\vec{J}\,dv$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\iiint u\,dv + \oiint\vec{P}\cdot\hat{n}\,ds = -\iiint\vec{E}\cdot\vec{J}\,dv} \quad \text{——} ⑨$$

* <u>Another form</u>:

From equation ⑧,

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) = -\vec{E}\cdot\vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \left(\vec{E}\cdot\epsilon\frac{\partial\vec{E}}{\partial t} + \vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t}\right) = -\vec{E}\cdot\vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) + \frac{1}{2}\mu\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = -\vec{E}\cdot\vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{1}{2}\cdot\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) + \frac{1}{2}\mu(\vec{B}\cdot\vec{H}) = -\vec{E}\cdot\vec{J}$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{1}{2}\frac{\partial}{\partial t}\left[\vec{B}\cdot\vec{H} + \vec{E}\cdot\vec{D}\right] = -\vec{E}\cdot\vec{J}} \quad \text{——} ⑩$$

⊖ Interpretation of the terms [equation (9)] : V.U - 2006, '07, '08

a) $\frac{\partial}{\partial t}\iiint u\,dv \Rightarrow$ Since u represents electromagnetic energy density, hence $\iiint u\,dv$ represents the electromagnetic energy within the whole volume. Thus $\frac{\partial}{\partial t}\iiint u\,dv$ represents the time rate of change of electromagnetic energy within certain volume.

b) $\oiint \vec{P}\cdot\hat{n}\,ds \Rightarrow$ Si $\vec{P} = \vec{E}\times\vec{H}$ has the dimension of energy flow per unit area per unit time. Hence $\oiint \vec{P}\cdot\hat{n}\,ds$ represents the rate of energy flow across a closed surface.

The vector $\vec{P} = \vec{E}\times\vec{H}$ is known as Poynting vector according to the name of the interpreter.

c) $-\iiint \vec{J}\cdot\vec{E}\,dv \Rightarrow -\iiint \vec{E}\cdot\vec{J}\,dv$ represents the rate of energy transferred into the electromagnetic field through the motion of free charge within the volume.

Which is explained as —

[Let us consider a charged particle within an electromagnetic field. Force acting on the particle, $\vec{F} = q\vec{E} + q(\vec{v}\times\vec{B})$

Rate of workdone by the e.m. field,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(\vec{F}\cdot\vec{r}) = \vec{F}\cdot\vec{v} = [q\vec{E} + q(\vec{v}\times\vec{B})]\cdot\vec{v}$$

$$= q\vec{E}\cdot\vec{v}. \quad \text{as} \quad \vec{v}\cdot\vec{v}\times\vec{B} = 0.$$

Rate of workdone against the e.m. field

$$-\frac{\partial w}{\partial t} = -q\vec{E}\cdot\vec{v}$$

Again $-\frac{\partial w}{\partial t} = -\sum n_i q_i \vec{E_i}\cdot\vec{v_i}$ for a group of particles.

$$\Rightarrow \quad -\frac{\partial w}{\partial t} = -\sum \vec{J_i}\cdot\vec{E_i} \quad \text{as } \vec{J_i} = \sum n_i q_i \vec{v_i}.$$

$$\Rightarrow \quad -\frac{\partial w}{\partial t} = -\vec{J}\cdot\vec{E}$$

i.e. $-\vec{J}\cdot\vec{E}$ represents the rate of energy transferred into the electromagnetic field.
Hence $-\iiint \vec{J}\cdot\vec{E}\,dv$ represents the rate of energy transferred into the whole volume of e.m. field.]

Differential form of Poynting theorem in non-conducting medium : V.U. 2006, 2008

In non-conducting medium $J=0$

We have Poynting theorem,

$$\frac{\partial}{\partial t}\iiint u\,dv + \oiint(\vec{E}\times\vec{H})\cdot\hat{n}\,ds = -\iiint \vec{J}\cdot\vec{E}\,dv.$$

Apply Gauss's theorem on the 2nd term of L.H.S,

$$\iiint \frac{\partial u}{\partial t}dv + \iiint \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = -\iiint \vec{J}\cdot\vec{E}\,dv.$$

Since $dv \neq 0$, $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\sigma\vec{E}\cdot\vec{E}$

For non-conducting medium $\sigma = 0$
and $\vec{E}\times\vec{H} = \vec{P}$, Poynting vect

$$\therefore \quad \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{P} = 0.$$

$$\therefore \quad \boxed{\vec{\nabla}\cdot\vec{P} + \frac{\partial u}{\partial t} = 0}$$ Differential form of Poyntin theorem.

Also, known as equation of contin

Poynting vector :

$\vec{P} = \vec{E} \times \vec{H}$, is known as Poynting vector according to the name of interpreter.

Poynting vector has the identical dimension of energy flow per unit area per unit time i.e power flux.

* Dimension : $[\vec{P}] = [E][H]$

$$= \frac{F}{q} \times \frac{q}{l \cdot t} \qquad \text{as } F = qE, \quad H = \frac{i}{l} = \frac{q}{t \cdot l}$$

$$\therefore [\vec{P}] = \left[\frac{MLT^{-2}}{LT}\right] = [MT^{-3}]$$

** S.I. unit of Poynting vector = Watt/m^2

*** The average value of Poynting vector represents the intensity of electromagnetic wave.

i.e $I = \langle p \rangle = \langle |\vec{E} \times \vec{H}| \rangle = \frac{1}{2} E_0 H_0$.

Plane electromagnetic wave propagating in free space :

Wave equation and velocity of e.m. wave and Max. equatio

Maxwell's electromagnetic equations are —

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{D} = \rho \\ &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\}$$

again : $\vec{B} = \mu\vec{H}$

$\vec{D} = \epsilon\vec{E}$

and $\vec{J} = \sigma\vec{E}$

We have, free space is characterised by
$$\left.\begin{aligned} \rho &= 0 \\ \sigma &= 0 \\ \epsilon &= \epsilon_0 \\ \mu &= \mu_0 \end{aligned}\right\}$$

Thus Maxwell's equations in free space is given by —

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 0 \quad \text{——} (1) \\ &\vec{\nabla}\cdot\vec{H} = 0 \quad \text{——} (2) \\ &\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial \vec{H}}{\partial t} \quad \text{——} (3) \\ &\vec{\nabla}\times\vec{H} = \epsilon_0\frac{\partial \vec{E}}{\partial t} \quad \text{——} (4) \end{aligned}}$$

Taking curl on both sides of equation (3), we get —

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0\,\vec{\nabla}\times\frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}).$$

$$\Rightarrow 0 - \nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}\left(\epsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$ using eqn. (1) and (4)

$$\therefore \boxed{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}} \quad \text{——} (5)$$

Similarly taking curl on both sides of equation (4), we get —

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \epsilon_0\,\vec{\nabla}\times\left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \cancel{\vec{\nabla}}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = \epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$. using eqn. (2) and (3)

$$\Rightarrow \boxed{\nabla^2\vec{H} = \mu_0\epsilon_0\frac{\partial^2\vec{H}}{\partial t^2}} \quad \text{——} (6)$$

Velocity of electromagnetic wave in free space:

Equation (5) and (6) are identical with 3D - wave equation

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{—— (7)}$$ Where c is the velocity of the wave.

Compairing we get,

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4 \times 3.14 \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \text{ m/sec}.$$

For a plane electromagnetic wave in free space show that the propagation vector ($\vec{K}$), the electric vector ($\vec{E}$) and the magnetic vectors ($\vec{H}$) are mutually perpendicular.

OR the e.m. wave is transverse in nature.

V.U - 1992, 1995, 1998, 2002.

Answer: We have the e.m. wave equations in free space as -

$$\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{—— (1)}$$

$$\nabla^2 \vec{H} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{—— (2)}.$$

Let us assume the plane wave solutions of above equations as -

$$\vec{E} = \vec{E_0}\, e^{i(-\omega t + \vec{K}\cdot\vec{r})} \quad \text{—— (3)}$$

$$\vec{H} = \vec{H_0}\, e^{i(-\omega t + \vec{K}\cdot\vec{r})} \quad \text{—— (4)}$$

Where $\vec{E_0}$ and $\vec{H_0}$ ⇒ field amplitudes, const.

ω ⇒ angular frequency.

$\vec{K}$ ⇒ propagation vector.

Substituting the solutions in Maxwell's 3rd equation,

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}.$$

$$\vec{\nabla} \times \left[\vec{E_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right] = -\mu_0 \frac{\partial}{\partial t} \left[\vec{H_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow (\cancel{\vec{\nabla} \times \vec{E_0}})^{0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} + \vec{\nabla}\left\{ e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right\} \times \vec{E_0}$$

$$= -\mu_0 \vec{H_0} (-i\omega) e^{i(\vec{K}\cdot\vec{r} - \omega t)}.$$

$$\Rightarrow i\vec{K} \times \vec{E_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} = i\omega\mu_0 \vec{H_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)}.$$

$$\Rightarrow i\vec{K} \times \vec{E} = i\omega\mu_0 \vec{H}.$$

$$\Rightarrow \boxed{\vec{K} \times \vec{E} = \mu_0 \omega \vec{H}} \text{ ——— } (5)$$

~~Taking curl~~ Substituting the solutions in Maxwell's 4th eqn.

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

$$\Rightarrow \vec{\nabla} \times \left[\vec{H_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right] = \epsilon_0 \frac{\partial}{\partial t} \left[\vec{E_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow (\cancel{\vec{\nabla} \times \vec{H_0}})^{0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} + \vec{\nabla}\left\{ e^{i(\vec{K}\cdot\vec{r} - \omega t)} \right\} \times \vec{H_0}$$

$$= \epsilon_0 \vec{E_0} (-i\omega) e^{i(\vec{K}\cdot\vec{r} - \omega t)}.$$

$$\Rightarrow i\vec{K} \times \vec{H_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)} = -i\omega\epsilon_0 \vec{E_0} e^{i(\vec{K}\cdot\vec{r} - \omega t)}$$

$$\Rightarrow \boxed{\vec{K} \times \vec{H} = -\epsilon_0 \omega \vec{E}} \text{ ——— } (6)$$

Eqn. (5) shows that $\vec{H}$ is perpendicular to both $\vec{K}$ and $\vec{E}$.
Eqn. (6) shows that $\vec{E}$ is perpendicular to both $\vec{K}$ and $\vec{H}$.

Hence $\vec{E}$, $\vec{H}$ and $\vec{K}$ are mutually perpendicular to each other i.e e.m. wave in free space is transverse in nature.

Ratio of magnitude of $\vec{E}$ and magnitude of $\vec{H}$:

We can prove, $\vec{K} \times \vec{E} = \mu_0 \omega \vec{H}$.

Taking magnitude, $KE = \mu_0 \omega H$

$$\Rightarrow \frac{E}{H} = \frac{\mu_0 \omega}{K} = \frac{\mu_0 \times 2\pi c/\lambda}{2\pi/\lambda} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\therefore Z_0 = \frac{E}{H} = \boxed{\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}}} = 376.6 \text{ ohm}.$$

Show That electrostatic and magnetostatic energy density are same in free space :

Answer : Electrostatic energy density $u_e = \frac{1}{2}\vec{D}\cdot\vec{E}$

$$= \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E}$$
$$= \frac{1}{2}\epsilon_0 E^2$$

Magnetostatic energy density $u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$

$$= \frac{1}{2}\mu_0 \vec{H}\cdot\vec{H}$$
$$= \frac{1}{2}\mu_0 H^2$$

$$\therefore \frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\times\left(\frac{E}{H}\right)^2 = \frac{\epsilon_0}{\mu_0}\times\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^2 = 1.$$

$$\therefore \boxed{u_e = u_m}$$

Show That the wave equation in free space can be written in the form $(\nabla^2+K^2)\vec{E}=0$, where K is the propagation const.

Answer :

Electromagnetic wave equation in free space,

$$\nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} \text{ ——— (1)}$$

Let us consider a plane wave which is propagating along Z-axis then $\vec{E}$ (electric field intensity) will vary only along Z-direction.

Then $\vec{E} = f(z,t) = \vec{E}_0 \cos(\omega t - Kz)$ say.

$$\Rightarrow \frac{\partial\vec{E}}{\partial t} = \cancel{\omega_0 \sin} - \omega\vec{E}_0 \sin(\omega t - Kz).$$

$$\Rightarrow \frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}_0\cos(\omega t - Kz)$$

$$\Rightarrow \frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E} = -K^2c^2\vec{E} \quad \text{as } \omega = \frac{2\pi c}{\lambda} = Kc$$

Substituting in equation (1),

$$\nabla^2\vec{E} = -\mu_0\epsilon_0 K^2c^2\vec{E}$$

$$\Rightarrow \underline{(\nabla^2+K^2)\vec{E}=0} \qquad \text{as } c^2 = \frac{1}{\mu_0\epsilon_0}$$

Show that $\vec{E}$ and $\vec{H}$ are in same phase in free space

Answer: Let us consider an electromagnetic wave which is propagating along z-direction. i.e E and H v along z direction only. $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$, $\frac{\partial}{\partial z} \neq 0$

i.e $E = E(z,t)$

and $H = H(z,t)$

Maxwell's 1st equation in free space

$\vec{\nabla}\cdot\vec{E} = 0$

$\Rightarrow \frac{\partial E_z}{\partial z} = 0$

$\Rightarrow \boxed{E_z = \text{const in space}}$

Maxwell's 2nd eqn. in free space

$\vec{\nabla}\cdot\vec{H} = 0$

$\Rightarrow \frac{\partial H_z}{\partial z} = 0$

$\Rightarrow \boxed{H_z = \text{Const in space}}$

Maxwell's 3rd equation in free space

$\vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$\Rightarrow \frac{\partial H_z}{\partial t} = 0$

$\Rightarrow \boxed{H_z = \text{const. in time}}$

as $(\vec{\nabla}\times\vec{E})_z = \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = 0$.

Maxwell's 4th eqn. in free space

$\vec{\nabla}\times\vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

$\Rightarrow \frac{\partial E_z}{\partial t} = 0$.

$\Rightarrow \boxed{E_z = \text{const. in time}}$

Hence: $E_z = 0$ and $H_z = 0$ (we may take).

Thus, $\left.\begin{array}{l}\vec{E} = \hat{i}E_x + \hat{j}E_y \\ \text{and } \vec{H} = \hat{i}H_x + \hat{j}H_y\end{array}\right\}$ Hence e.m. wave is transverse in nature.

If c be the velocity of e.m. wave then the component—

$E_x = E_0 \cos\omega(t - z/c)$ —— ①

and $H_y = H_0 \cos\omega(t - z/c)$.

$\Rightarrow H_y = \frac{1}{377}\cdot E_0 \cos\omega(t - z/c)$ —— ② as $\frac{E_0}{H_0} = 377$.

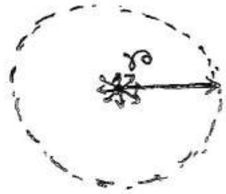
From equation ① and ②, we see that

$\vec{E}$ and $\vec{H}$ are in same phase.

Problem (1) : An observer is at a distance 1 m. from a point source whose power input is 1000 watt. Calculate the electric and magnetic fields as observed by the observer.

V.U - 1997

Solution : The energy emitted by the source spreads over the area of a sphere of radius $r = 1$ m.



So, amount of energy crossing per unit area per unit time at a distance of 1 m. i.e Poynting vector,

$$P = \frac{\text{Power}}{\text{area}} = \frac{1000 \text{ watt}}{4\pi (1)^2 m^2} = \frac{1000}{4\pi} \text{ watt}/m^2.$$

Again average flow of energy = Average value of Poynting vector.

$$= \langle P \rangle = \frac{1}{2} E_0 H_0$$

$$= \frac{1}{2} E_0 \times \frac{E_0}{377} \quad \text{as } \frac{E_0}{H_0} = 377$$

$$= \frac{E_0^2}{2 \times 377}$$

Thus, $\frac{E_0^2}{2 \times 377} = \frac{1000}{4\pi}$

$$\Rightarrow E_0^2 = \frac{2 \times 377 \times 1000}{4 \times 3.14}$$

$$\Rightarrow E_0 = \left(\frac{2 \times 377 \times 1000}{4 \times 3.14} \right)^{1/2} = \underline{89.5 \text{ Volt}/m}.$$

And $H_0 = \frac{E_0}{377} = \frac{89.5}{377} = \underline{0.24 \text{ Ampere}/m}.$

N.B : Average value of Poynting vector,

$$\vec{P} = \vec{E} \times \vec{H}$$

$$\Rightarrow P = EH \quad \text{as } \vec{E} \perp r \; \vec{H}$$

$$\therefore \langle P \rangle = \langle E_0 \sin(\omega t - \vec{K}\cdot\vec{r}) \rangle \langle H_0 \sin(\omega t - \vec{K}\cdot\vec{r}) \rangle$$

$$= E_0 H_0 \langle \sin^2(\omega t - \vec{K}\cdot\vec{r}) \rangle$$

$$= E_0 H_0 \times \frac{1}{2}$$

$$= \frac{1}{2} E_0 H_0$$

Problem (2) : Find the magnetic field in air of a distance of 100 cm. from a radiator of power 10 KW.
V.U-1995

Solution : Energy crossing per unit area per unit time at a distance of r from source $= \frac{\text{Power}}{\text{area}} = \frac{10\times1000 \text{ watt}}{4\pi(1)^2 m^2}$

$$= \frac{10000}{4\pi} \text{ watt}/m^2.$$

Average Poynting vector $= \langle P\rangle = \frac{1}{2}E_0H_0 = \frac{1}{2}\times377\, H_0^2$.

$$\therefore \quad \frac{377}{2}H_0^2 = \frac{1000}{4\pi}$$

$$\Rightarrow \quad H_0 = \left(\frac{2000}{377\times4\times3.14}\right)^{1/2} = \underline{2.05 \text{ Ampere}/m.}$$

Problem (3) : The intensity of sun light reaching earth's surface is 2 cal per square cm/min. Calculate the electric field in volt/meter of the coming sunlight.
V.U-1996

Solution : The intensity of sunlight reaching earth's surface i.e energy falling per unit area per unit time.

$$= 2 \text{ Cal}/cm^2/min.$$

$$= \frac{2\times4.2\times10^4}{60} \text{ J}/m^2/sec.$$

Average energy flow = Average value of Poynting vector.

$$\langle P\rangle = \frac{1}{2}E_0H_0$$

$$= \frac{1}{2}\times E_0\times\frac{E_0}{377}$$

$$= \frac{E_0^2}{2\times377}.$$

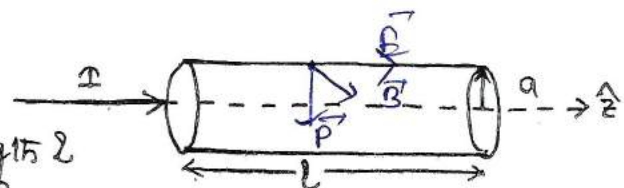
$$\therefore \quad \frac{E_0^2}{377} = \frac{2\times4.2\times10^4}{60}$$

$$\Rightarrow \quad E_0 = \left(\frac{2\times4.2\times10^4\times377}{60}\right)^{1/2} = \underline{1027.42 \text{ v}/m.}$$

Problem (4): V.U-1992, 1994, 1996, 2006

Find the value of $\vec{E}$ and $\vec{H}$ on the surface of a wire carrying a current. Show by computing the Poynting vector that it represents a flow of energy into the wire.

Solution:



Let us consider a wire of length l and radius a, carrying a current I.

If V be the p.d. applied across the ends of the wire, then electric field along the direction of current $\boxed{\vec{E} = \frac{V}{l}\hat{z} \text{ volt/m}}$.

According to Ampere's circuital law,

$$\oint \vec{H}\cdot d\vec{l} = I$$

$$\Rightarrow H \cdot 2\pi a = I$$

$$\Rightarrow \boxed{\vec{H} = \frac{I}{2\pi a}\hat{x} \text{ Amp/m}}$$, as it is directed tangent to the surface.

So the Poynting vector $\vec{P} = \vec{E}\times\vec{H} = EH\,\hat{n}$

$$= \frac{V}{l}\cdot\frac{I}{2\pi a}\cdot\hat{n}$$

$$= \frac{VI}{2\pi a l}\hat{n}$$ (directed along inward direction)

Thus the rate of flow of energy through the surface of conductor,

$$\oiint \vec{P}\cdot\hat{n}\,ds = \oiint \frac{VI}{2\pi a l}\hat{n}\cdot\hat{n}\,ds$$

$$= \frac{VI}{2\pi a l}\times 2\pi a l$$ as $\oiint ds$ = area of curved surface of conductor $= 2\pi a l$.

$$= VI$$

Again, rate of flow of energy into the wire,

$$\iiint \vec{J}\cdot\vec{E}\,dv = \iiint \sigma\vec{E}\cdot\vec{E}\,dv = \iiint \sigma E^2 dv$$

$$= \sigma\left(\frac{V}{l}\right)^2 \iiint dv$$

$$= \sigma\frac{V^2}{l^2}\times\pi a^2 l = V^2\times\frac{\sigma A}{l} = V^2/R = VI.$$

Propagation of electromagnetic wave through dielectric medium

An ideal dielectric medium is characterised by,

$$\mu \neq 0$$
$$\epsilon \neq 0$$
$$\rho = 0$$
$$\sigma = 0$$

Again $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.

Hence Maxwell's equations in a dielectric medium are ....

$$\vec{\nabla} \cdot \epsilon \vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{E} = 0 \quad \text{——— (1)}$$

$$\vec{\nabla} \cdot \mu \vec{H} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{H} = 0 \quad \text{——— (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\mu \vec{H}) \quad \Rightarrow \quad \vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{——— (3)}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \frac{\partial}{\partial t}(\epsilon \vec{E}) \quad \Rightarrow \quad \vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{——— (4)}$$

Taking curl on both sides of equation (3), we get,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\Rightarrow \nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left(\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{——— (5)}$$

Taking curl on both sides of equation (4), we get,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H} = \epsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H} = \epsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 \vec{H} = \mu \epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{——— (6)}$$

Compairing equation (5) and (6) with general wave eqn. $\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$ we get velocity of electromagnetic wave in dielectric medium.

$$\boxed{c = \frac{1}{\sqrt{\mu \epsilon}}}$$

$$\Rightarrow (\cancel{\vec{\nabla}\times\vec{E}_0})^{0} e^{i(\vec{K}\cdot\vec{r}-\omega t)} + \vec{\nabla}\{e^{i(\vec{K}\cdot\vec{r}-\omega t)}\}\times\vec{E}_0 = -\mu\vec{H}_0(-i\omega)e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$

$$\Rightarrow i\vec{K}\times\vec{E}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)} = i\omega\mu_0\vec{H}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$

$$\Rightarrow i\vec{K}\times\vec{E} = i\omega\mu\vec{H}.$$

$$\Rightarrow \boxed{\vec{K}\times\vec{E} = \omega\mu\vec{H}} \text{ ——— } (5)$$

Substituting the solutions in Maxwell's 4th equation,

$$\vec{\nabla}\times\vec{H} = \epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times[\vec{H}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}] = \epsilon\frac{\partial}{\partial t}[\vec{E}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}]$$

$$\Rightarrow (\vec{\nabla}\times\vec{H}_0)e^{i(\vec{K}\cdot\vec{r}-\omega t)} + \vec{\nabla}\{e^{i(\vec{K}\cdot\vec{r}-\omega t)}\}\times\vec{H}_0$$
$$= \epsilon_0\vec{E}_0(-i\omega)e^{i(\vec{K}\cdot\vec{r}-\omega t)}.$$

$$\Rightarrow i\vec{K}\times\vec{H}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)} = -i\omega\epsilon\vec{E}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}.$$

$$\Rightarrow i\vec{K}\times\vec{H} = -i\omega\epsilon\vec{E}.$$

$$\Rightarrow \boxed{\vec{K}\times\vec{H} = -\omega\epsilon\vec{E}} \text{ ——— } (6)$$

Equation (5) shows that $\vec{H}$ is perpendicular to both $\vec{K}$ and $\vec{E}$.
Equation (6) shows that $\vec{E}$ is perpendicular to both $\vec{K}$ and $\vec{H}$.
Hence $\vec{E}$, $\vec{H}$ and $\vec{K}$ are mutually perpendicular to each other.

Show that in a linear dielectric medium, ~~electrostatic~~ electromagnetic energy density is equally shared between electric and magnetic fields.

V.U – 2005, 2006, 2007

Electrostatic energy density $u_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\epsilon_0\vec{E}\cdot\vec{E} = \frac{1}{2}\epsilon E^2$.
Magnetostatic energy density $u_m = \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{1}{2}\mu\vec{H}\cdot\vec{H} = \frac{1}{2}\mu H^2$

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\epsilon}{\mu}\left(\frac{E}{H}\right)^2 = \frac{\epsilon}{\mu}\left(\sqrt{\frac{\mu}{\epsilon}}\right)^2 = 1.$$

$$\therefore \boxed{u_e = u_m}$$

Propagation of electromagnetic wave through conducting medium :

A conducting medium is characterised by,

permeability $= \mu$, permittivity $= \epsilon$ and conductivity $= \sigma$

In this case, $\vec{D} = \epsilon \vec{E}$

$\vec{B} = \mu \vec{H}$

$\vec{J} = \sigma \vec{E}$

and $\rho = 0$.

Hence Maxwell's equations in a conducting medium,

$$\vec{\nabla} \cdot \epsilon \vec{E} = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0 \quad \text{——— (1)}$$

$$\vec{\nabla} \cdot \mu \vec{H} = 0 \Rightarrow \vec{\nabla} \cdot \vec{H} = 0 \quad \text{——— (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\mu \vec{H}) \Rightarrow \vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{——— (3)}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \frac{\partial}{\partial t}(\epsilon \vec{E}) \Rightarrow \vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{——— (4)}$$

Taking curl on both sides of equation (3), we get —

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\Rightarrow -\nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right) \quad \text{using eqn. (1) and (4).}$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \text{——— (5)}$$

Taking curl on both sides of equation (4), we get —

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H} = \sigma(\vec{\nabla} \times \vec{E}) + \epsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

$$\Rightarrow -\nabla^2 \vec{H} = -\sigma \mu \frac{\partial \vec{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 \vec{H} = \mu \sigma \frac{\partial \vec{H}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \quad \text{——— (6)}$$

Equation (5) and (6) represents electromagnetic wave equation satisfied by electric and magnetic field respectively inside a conducting medium.

Attenuation of electromagnetic wave inside a conducting med

Question: Show that electromagnetic wave is damped inside a conducting medium.

Electromagnetic wave equations satisfied by electric and magnetic field inside a conducting medium, are given by,

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{------- (1)}$$

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{------- (2)}$$

Let us assume the plane wave eq. solutions of above equations are,

$$\vec{E} = \vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{-------- (3)}$$

$$\vec{H} = \vec{H_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{------- (4)}$$

Substituting above solutions in equation (1), we get-

$$-K^2\vec{E} = \mu\sigma(-i\omega\vec{E}) + \mu\epsilon(-\omega^2\vec{E})$$

$$\Rightarrow K^2\vec{E} = i\mu\sigma\omega\vec{E} + \mu\epsilon\omega^2\vec{E} \qquad \text{as } \vec{E} \neq 0.$$

$$\Rightarrow K^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \quad \text{------ (5)}$$

Similarly, substituting above solutions in equation (2), we get similar equation as equation (5).

Let us assume $K = \alpha + i\beta$ ----- (6)

Evaluation of α and β

$$K^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \quad \text{------ (1)}$$

Let $K = \alpha + i\beta$ --------- (2)

Squaring, $K^2 = (\alpha^2 - \beta^2) + i\,2\alpha\beta$ ----- (3)

Comparing equation (1) and (3), we get—

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2 \quad \text{----- (4)}$$

$$2\alpha\beta = \mu\sigma\omega \quad \text{----- (5)}$$

Now, $(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$

$= \{\mu\epsilon\omega^2\}^2 + \{\mu\sigma\omega\}^2$

$= (\mu\epsilon\omega^2)^2\left[1 + \frac{\sigma^2}{\epsilon^2\omega^2}\right]$

$\therefore\ \alpha^2+\beta^2 = \mu\epsilon\omega^2\left(1+\frac{\sigma^2}{\epsilon^2\omega^2}\right)^{1/2}$ ------- (6)

Adding eqn. (4) and (6),

$$\boxed{\alpha = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\left(\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1\right)^{1/2}} \quad ----- (7)$$

Subtracting eqn (4) and (6)

$$\boxed{\beta = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\left(\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}-1\right)^{1/2}} \quad ----- (8)$$

Case I : For poor conductor :

For poor conductor σ is small, hence $\frac{\sigma}{\epsilon\omega} \ll 1$ <say>

Thus, $\alpha = \sqrt{\mu\epsilon\omega^2/2} \times \sqrt{2} = \boxed{\sqrt{\mu\epsilon\omega^2}}$

and $\beta = \sqrt{\mu\epsilon\omega^2/2}\left(1+\frac{\sigma^2}{2\epsilon^2\omega^2}-1\right)^{1/2} = \boxed{\sqrt{\frac{\mu\sigma^2}{4\epsilon}}}$

Case II : For good conductor :

For good conductor σ is large, hence $\frac{\sigma}{\epsilon\omega} \gg 1$ <say>

Thus, $\alpha = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\frac{\mu\sigma\omega}{2}}}$

and $\beta = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\frac{\mu\sigma\omega}{2}}}$

Hence the electric and magnetic field vectors can be written as,

$\vec{E} = \vec{E}_0\, e^{i(k\hat{n}\cdot\vec{r}-\omega t)}$ where $\hat{n}$ is the unit vector along the direction of propagation.

$= \vec{E}_0\, e^{i\{(\alpha+i\beta)(\hat{n}\cdot\vec{r})-\omega t\}}$

$= \vec{E}_0\, e^{-\beta\hat{n}\cdot\vec{r}} \cdot e^{i\{\alpha(\hat{n}\cdot\vec{r})-\omega t\}}$ ------- (7)

Similarly, $\vec{H} = \vec{H}_0\, e^{-\beta\hat{n}\cdot\vec{r}} \cdot e^{i\{\alpha(\hat{n}\cdot\vec{r})-\omega t\}}$ ------- (8)

Equation ⑦ and ⑧ represent that the field amplitudes are spatially attenuated due to presence of the term $e^{-\beta \hat{n} \cdot \vec{r}}$ i.e electromagnetic wave is damped inside the conducting medium

The quantity β is called attenuation factor.
and α is called phase factor.

Define skin depth or penetration depth. V.U. 2008

Skin depth: It is defined as the distance traversed by an electromagnetic wave in a conducting medium at which the field amplitude decays to 1/e th time of its initial / maximum value.

Derivation: The field amplitudes are $\vec{E_0}\, e^{-\beta \hat{n} \cdot \vec{r}}$ (electric)
and $\vec{H_0}\, e^{-\beta \hat{n} \cdot \vec{r}}$ (magnetic)

Where $\vec{E_0}$ and $\vec{H_0}$ are maximum value of field amp. (r=

If δ be the skin depth, then at $r = \delta$, $A = \frac{E_0}{e}$ or $A = \frac{H}{e}$

i.e $$E_0\, e^{-\beta\delta} = \frac{E_0}{e}$$

$$\Rightarrow e^{-\beta\delta} = e^{-1}$$

$$\Rightarrow \boxed{\delta = \frac{1}{\beta}}$$

Skin depth for bad conductor, $\delta = \sqrt{\frac{4\epsilon}{\mu\sigma^2}}$

Skin depth for good conductor, $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$

Attenuation of e.m. wave inside a conducting medium shown graphically,