Upper Sum & Lower Sum

Partition of a set:

Partition of a set is defined as "a collection of distinct subset of a given set. The union of subsets must be equal to the entire original set

For Example: One possible partition of $\{1, 2, 3, 4, 5, 6\}$ is $\{1, 3\}, \{2\}, \{4, 5, 6\}$.

Partition:

Let $[a, b]$ be a given closed interval. By a partition P of $[a, b]$, we mean a finite set of points $x_0, x_1, x_2, \cdots, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

The partition P consists of $n+1$ points. Clearly any number of partitions of $[a, b]$ can be considered.

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{i-1}, x_i], \cdots [x_{n-1}, x_n]$$

are the n sub-intervals of $[a, b]$

Now we denote Δx_i as ith sub-interval $[x_{i-1}, x_i]$ length

- $\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, 3 \cdots, n$

Note: $P = \{x_0, x_1, x_2, \cdots, x_n\} \subseteq [a, b]$

Let f be a bounded real valued function on $[a, b]$. Evidently f is bounded on each sub-interval corresponding to each partition P. Let M_i, m_i be the bounds (Supremum & infimum) of f in Δx_i

Now,

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

are called the upper and Lower (Darboux) Sum of f corresponding to the partion P.

If M, m are the bounds of f in $[a, b]$ we have.

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \Delta x_i \leq m_i \Delta x_i \leq M_i \Delta x_i \leq M \Delta x_i$$

$$\Rightarrow m \sum_{i=1}^{n} \Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq M \sum_{i=1}^{n} \Delta x_i$$

$$\Rightarrow m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \cdots\cdots (1)$$

Now each partition gives rise to a pair of sums the upper & the lower sums.

The above inequality shows that both these sets are bounded and so each set has the supremum and infimum.

The infimum of the set of upper sum is called upper integral and denoted by $\overline{\int_a^b} f\,dx = \inf U(P,f)$

The suprimum of the set of Lower sum is called the Lower integral over $[a,b]$ and denoted by

$$\underline{\int_a^b} f\,dx = \sup L(P,f).$$

Definition: (Darboux's condition of integrability)

When the two integrals are equal i.e.

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx$$

Then we say that f is Rieman Integrable over $[a,b]$

Now from (1)

$$m(b-a) \leq \int_a^b f\,dx \leq M(b-a), \quad b \geq a$$

Thus the upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function.

Ex: Show that a constant function k is integrable and $\int_a^b k\,dx = k(b-a)$

For any partition P of the interval $[a,b]$ we have

$$L(P,f) = k\Delta x_1 + k\Delta x_2 + \cdots + k\Delta x_n$$

$$= k(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n)$$

$$= k(x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1})$$

$$= k(b-a)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f) = k(b-a)$$

$$\overline{\int_a^b} f\,dx = \inf U(P,f) = \inf(k\Delta x_1 + k\Delta x_2 + \cdots + k\Delta x_n)$$

$$= k(b-a)$$

Thus $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = k(b-a)$

Which implies that the funⁿ $f(x) = k$ is integrable and $\int_a^b k\,dx = k(b-a)$

Ex2 Show that the function f defined by

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

is not integrable on any interval

Let us consider a partition P of an interval $[a,b]$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= 1\cdot\Delta x_1 + 1\cdot\Delta x_2 + 1\cdot\Delta x_3 + \cdots + 1\cdot\Delta x_n$$

$$= x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \cdots + x_n - x_{n-1}$$

$$= b-a$$

$$\overline{\int_a^b} f\,dx = \inf U(P,f) = b-a$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f) = \sup\{0\Delta x_1 + 0\Delta x_2 + \cdots + 0\Delta x_n\} = 0$$

Thus $\overline{\int_a^b} f\,dx \neq \underline{\int_a^b} f\,dx$

Hence the function f is not integrable

Ex3 Show that $f(x) = x^2$ is integrable on any interval $[0, k]$

Let us consider the partition P of $[0, k]$ obtained by dividing the sub intervals into n equal parts. Thus $[0, \frac{k}{n}, \frac{2k}{n}, \dots, \frac{nk}{n}]$ is the Partition of P $\left[(i-1)\frac{k}{n}\right]^2$ and $\left[i\frac{k}{n}\right]^2$ are the lower and upper bounds of the function in Δx_i, and the length of each subinterval is $\frac{k}{n}$

$$\therefore U(P, x^2) = \left(\frac{1k}{n}\right)^2 \cdot \left(\frac{k}{n} - 0\right) + \left(\frac{2k}{n}\right)^2 \cdot \left(\frac{2k}{n} - \frac{k}{n}\right) + \cdots + \left(\frac{nk}{n}\right)^2 \cdot \left(\frac{nk}{n} - \frac{(n-1)k}{n}\right)$$

$$= \frac{k^3}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

$$= \frac{k^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{k^3}{6} \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

and

$$L(P, x^2) = \left(\frac{0k}{n}\right)^2\left(\frac{k}{n} - 0\right) + \left(\frac{1k}{n}\right)^2\left(\frac{2k}{n} - \frac{k}{n}\right) + \left(\frac{2k}{n}\right)^2 \cdot \left(\frac{3k}{n} - \frac{2k}{n}\right) + \cdots + \left(\frac{(n-1)k}{n}\right)^2 \cdot \left(\frac{nk}{n} - \frac{(n-1)k}{n}\right)$$

$$= \frac{k^3}{n^3}\left(0^2 + 1^2 + 2^2 + \cdots + (n-1)^2\right)$$

$$= \frac{k^3}{n^3} \frac{(n-1)(n)(2(n-1)+1)}{6}$$

$$L(P, x^2) = \frac{k^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\therefore \operatorname{Inf} U(P, x^2) = \frac{k^3}{3} = \operatorname{Sup} L(P, x^2)$$

Hence, the function $f(x) = x^2$ is integrable

and $\int_0^k x^2\,dx = \frac{k^3}{3}$

Note:

i) $\int_a^b f\,dx$ exists, implies that the function f is bounded and integrable over $[a, b]$

ii) We have introduced the concept of integrability of a function subject to two very important Limitation, viz (a) the funⁿ is bounded
(b) the interval is finite

iii) $b > a$,
$$m(b-a) \leq L(P, f) \leq \int_a^b f\,dx \leq U(P, f) \leq M(b-a)$$

iv) Since the upper integral is the greatest lower bound of the set of upper sum, therefore corresponding to any $\epsilon > 0$ $\exists$ an upper sum (or $\exists$ a partition P_1) Such that

$$U(P_1, f) < \overline{\int_a^b} f\,dx + \varepsilon_1$$

Since the lower integral is the Least upper bound of the set of Lower sum, corresponding to any $\varepsilon_2 > 0$ $\exists$ a Lower sum (or a partition P_2) Such that

$$L(P_2, f) > \underline{\int_a^b} f\,dx - \varepsilon_2$$

v) $U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$

$$= \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$

$M_i - m_i$ being the oscilation of f in the sub-interval Δx_i,

$U(P,f) - L(P,f)$ is called the oscilatory sum denoted by $\omega(P,f)$ and is non-negative.

Refinement of Partition :

For any partition P, the length of the largest sub-interval is called the norm or mesh of the partition and is denoted by $\|P\|$ or $\mu(P)$

$\therefore \mu(P) = \max \Delta x_i \quad (1 \leq i \leq n)$

A partition P^* is said to be a refinement of P if $P^* \supseteq P$ i.e. every point of P is a point of P^*

We also say that P^* refines P or P^* is finer than P.

Theorem:

If P^* is a refinement of a partition P, then for a bounded function f,

(i) $L(P^*, f) \geqslant L(P, f)$ and

(ii) $U(P^*, f) \leq U(P, f)$.

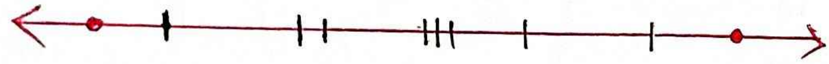
Corollary: If a refinement P^* of P contain p points more than P, and $|f(x)| \leq k$ for all $x \in [a, b]$ then

$$L(P, f) \leq L(P^*, f) \leq L(P, f) + 2pk\mu$$

$$U(P, f) \geq U(P^*, f) \geq U(P, f) - 2pk\mu$$

Ex: $\{1, \frac{3}{2}, 2, 3, 4, \frac{9}{2}, 5\}$ is a refinement of $\{1, \frac{3}{2}, 2, 4, 5\}$

P



$Q = P^*$

Darboux's Theorem:

If f is a bounded function on $[a, b]$, then for every $\varepsilon > 0$, there corresponding $\delta > 0$ such that

(i) $U(P, f) < \overline{\int_a^b} f \, dx + \varepsilon$

(ii) $L(P, f) > \underline{\int_a^b} f \, dx - \varepsilon$

for every partition P of $[a, b]$ with norm $\mu(P) < \delta$.

Conditions of Integrability

We have stated earlier that a bounded function is said to be integrable when the upper integral and Lower integral are equal

We now formalise and give the necessary and sufficient condition for the integrability of a function in two form

Theorem (First form):

A necessary and sufficient condition for the integrability of a bounded function f is that to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a\ b]$ with norm $\mu(P) < \delta$,

$$U(P, f) - L(P, f) < \varepsilon$$

Proof: The condition is necessary.

The bounded function f is integrable

$$\therefore \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx$$

Let ε be any positive number. By Darboux's Theorem $\exists$ $\delta > 0$ s.t. for every partition P with norm $\mu(P) < \delta$

$$U(P,f) < \overline{\int_a^b} f\,dx + \frac{1}{2}\varepsilon \quad \cdots (i)$$

$$L(P,f) > \underline{\int_a^b} f\,dx - \frac{1}{2}\varepsilon \quad \cdots (ii)$$

or $$-L(P,f) < -\underline{\int_a^b} f\,dx + \frac{\varepsilon}{2} \quad \cdots (iii)$$

adding (i) & (iii) we get

$$U(P,f) - L(P,f) < \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx + \varepsilon$$

$$\Rightarrow U(P,f) - L(P,f) < \varepsilon \quad \left[\because \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx\right]$$

for every partition P with norm $\mu(P) < \delta$

The condition is sufficient. Let $\varepsilon > 0$ be any positive no. For any partition P with norm $\mu(P) < \delta$

$$U(P,f) - L(P,f) < \varepsilon$$

Also for any partition P, we know that

$$L(P,f) \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le U(P,f)$$

$$\Rightarrow \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \le U(P,f) - L(P,f) < \varepsilon$$

Since ε is an arbitrary positive no.

therefore we see that a non-negative no

is less than every positive no.

Hence it must be equal to zero

$$\Rightarrow \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx = 0$$

$$\Rightarrow \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$$

Hence f is integrable

Theorem: (Second form)

A bounded funn f is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ s.t.

$$U(P,f) - L(P,f) < \varepsilon$$

The condition is necessary. Suppose the funn f is integrable, so that $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \int_a^b f\,dx$

Let ε be positive no.
Since the upper and Lower integrals are the infimum and suprimum, respectively, of the upper and Lower sums, therefore $\exists$ partition P_1 & P_2 such that

$$U(P_1, f) < \overline{\int_a^b} f\,dx + \frac{1}{2}\varepsilon = \int_a^b f\,dx + \frac{1}{2}\varepsilon$$

$$L(P_2, f) > \underline{\int_a^b} f\,dx - \frac{1}{2}\varepsilon = \int_a^b f\,dx - \frac{\varepsilon}{2}$$

Let P be the common partition of P_1 & P_2 i.e. $P = P_1 \cup P_2$

$$\therefore U(P, f) \leq U(P_1, f) < \int_a^b f\,dx + \frac{1}{2}\varepsilon < L(P_2, f) + \frac{\varepsilon}{2} < L(P, f) + \frac{\varepsilon}{2}$$

Thus $\exists$ a partition P s.t.

$$U(P, f) - L(P, f) < \varepsilon$$

The condition is sufficient. Let $\varepsilon > 0$
Let P be a partition for which $U(P, f) - L(P, f) < \varepsilon$
For any partition P we know that

$$L(P, f) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq U(P, f)$$

$$\therefore \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq U(P, f) - L(P, f) < \varepsilon$$

Since ε is an arbitrary positive no. therefore we see that a non-negative no. is less than every +ve no. Hence it must be equal to zero, $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$
So f is integrable