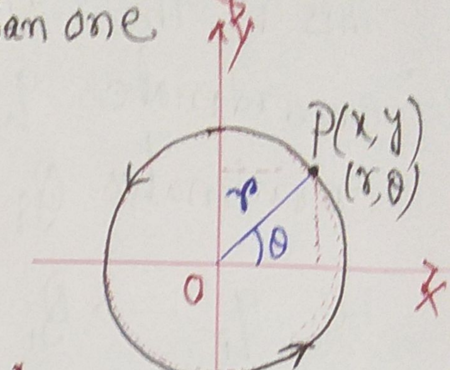
Canonical Transformation & Generating function.

A given system can be described by more than one set of generalized co-ordinates and their corresponding generalized momenta.



Partical move on motion under some central force i.e. gravitational force or electrostatic force

We choose a set of generalized co-ordinates which is more convenient for the solution of our problem.

i.e. to discuss the motion of a partical in a plane under central force we may use either $q_1 = x,\ q_2 = y$ or $q_1 = r,\ q_2 = \theta$

Here (r, θ) set is more convenient as compaired to (x, y) set because the polar co-ordinate (r, θ) is in cyclic nature but the co-ordinate (x, y) is not in cyclic nature in general.

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$\therefore x^2 + y^2 = r^2$$

$$\Rightarrow \sqrt{x^2+y^2} = r \quad \cdots \text{(i)}$$

Now $$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\Rightarrow \tan\theta = \frac{y}{x}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \cdots \text{(ii)}$$

Equation (i) & (ii) shows the transformation of cartesian co-ordinate (x, y) to polar co-ordinate (r, θ).

Point transformation:

This is the transformation from one set of generalised co-ordinates q_i to another set of generalised co-ordinates Q_i which is called a point transformation.

i.e. $q_i \longrightarrow Q_i$ Such that $Q_i = Q_i(q_i, t)$, $i = 1, 2, \dots n$

$$= Q_i(q_1, q_2, \dots, q_n, t)$$

The point transformation is also called as transformation of configuration space" which spanned by these n q-co-ordinates. Since the configuration space provides the information only about position co-ordinates q_i

Canonical Transformation:

To easy in solution of several problems in classical mechanics we may need to change one set of position & momentum co-ordinates to an another set of position co-ordinates & momentum co-ordinates.

Let q_i and p_i are old position & momentum coordinates and Q_i & P_i are new position & momentum co-ordinates. Then according to canonical transformation these co-ordinates are related to each other by following equations,

$$Q_i = Q_i(q_i, p_i, t) = Q_i(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n, t)$$

$$P_i = P_i(q_i, p_i, t) = P_i(q_1, q_2, \cdots q_n, p_1, p_2, \cdots p_n, t).$$

Canonical transformation are also called Transformation of phase space. Since phase space provides information about position co-ordinates as well as generalised momenta.

These transformation are characterised by property that they leave Lagrange's & Hamiltonian equations of motions unchanged.

$$L = L(q_i, \dot{q}_i, t), \quad H = H(q_i, p_i, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad H = \sum_i p_i \dot{q}_i - L$$

$$\Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

In new co-ordinates system

$$L' = L'(Q_i, \dot{Q}_i, t), \quad H' = H'(Q_i, P_i, t)$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}_i}\right) - \frac{\partial L'}{\partial Q_i} = 0, \quad H' = \sum_i P_i \dot{Q}_i - L'$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i}, \quad \dot{Q}_i = \frac{\partial H'}{\partial P_i}$$

As q_i, p_i are always canonical so they satisfy Hamilton's variational principle,

i.e. $\delta \int_{t_0}^{t} L\,dt = 0$ or $\delta \int_{t_0}^{t_1} (\sum p_i \dot{q}_i - H)\,dt = 0 \cdots\cdots$ (i)

As Q_i, P_i are also canonical, they must satisfy Hamilton's variational principle,

i.e. $\delta \int_{t_0}^{t} L'\,dt = 0$ or $\delta \int_{t_0}^{t_1} (\sum P_i \dot{Q}_i - H')\,dt = 0 \cdots\cdots$ (ii)

Equation (i) & (ii) does not mean that the integrands of both integrads are equal.

Equation (i) & (ii) are satisfied if their integrands are connected by the relation $(\sum p_i \dot{q}_i - H) - (\sum P_i \dot{Q}_i - H') = \frac{dF}{dt}$

$\cdots$ (iii)

Here F is called Generating function and is a function of q_i, p_i, Q_i, P_i, t.

There are basically 4 forms of F

(a) $F_1 = F_1(q_i, Q_i, t)$

(b) $F_2 = F_2(q_i, P_i, t)$

(c) $F_3 = F_3(p_i, Q_i, t)$

(d) $F_4 = F_4(p_i, P_i, t)$

out of this 4 forms of F choice of our particular form depends on the problem under consideration.

Generating functions:

Case-I Let $F_1 = F_1(q_i, Q_i, t)$

Diff w.r.t 't'

$$\frac{dF_1}{dt} = \sum \frac{\partial F_1}{\partial q_i} \cdot \frac{dq_i}{dt} + \sum \frac{\partial F_1}{\partial Q_i} \cdot \frac{dQ_i}{dt} + \frac{\partial F_1}{\partial t}$$

$$= \sum_i \frac{\partial F_1}{\partial q_i} \cdot \dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \quad \cdots (1)$$

as $\frac{dF}{dt} = \sum p_i \dot{q}_i - H - \sum P_i \dot{Q}_i + H' \quad \cdots (ii)$

Comparing (i) & (ii) $\quad p_i = \frac{\partial F_1}{\partial q_i}, \; P_i = -\frac{\partial F_1}{\partial Q_i}, \; H' - H = \frac{\partial F_1}{\partial t}$

Case-II Let $F_2 = F_2(q_i, P_i, t)$

Now $F_2(q_i, P_i, t)$ can be obtained from $F_1(q_i, Q_i, t)$ by using Legendre transformation

$$f(x, y) \xrightarrow{L.T} g(u, y)$$

$$g(u, y) = f(x, y) - \sum_i u_i x_i$$

$$u = \frac{\partial f}{\partial x}$$

$$F_1(q_i, Q_i, t) \xrightarrow{L.T} F_2(q_i, P_i, t)$$

$$F_2(q_i, P_i, t) = F_1(q_i, Q_i, t) - \sum P_i Q_i$$

But as $P_i = -\frac{\partial F_1}{\partial Q_i}$

$$\therefore u = -P_i$$

$$\therefore F_2(q_i, P_i, t) = F_1(q_i, Q_i, t) + \sum P_i Q_i$$

$$F_1(q_i, Q_i, t) = F_2(q_i, P_i, t) - \sum P_i Q_i$$

as $\frac{dF_1}{dt} = \sum p_i \dot{q}_i - H - \left(\sum P_i \dot{Q}_i - H'\right)$

$$\Rightarrow \sum p_i \dot{q}_i - H - \left(\sum P_i \dot{Q}_i - H'\right) = \frac{d}{dt}\left(F_2(q_i, P_i, t) - \sum P_i Q_i\right)$$

$$= \frac{dF_2(q_i, P_i, t)}{dt} - \frac{d}{dt}\left(\sum P_i Q_i\right)$$

$$\Rightarrow \sum p_i \dot{q}_i - H - \sum P_i \dot{Q}_i + H' = \sum \frac{\partial F_2}{\partial q_i} \cdot \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} \cdot 1$$

$$- \sum P_i \dot{Q}_i - \sum \dot{P}_i Q_i$$

$$\Rightarrow \sum \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} = \sum p_i \dot{q}_i + \sum \dot{P}_i Q_i + H' - H$$

Comparing both sides

$$p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}, \quad H' - H = \frac{\partial F_2}{\partial t}$$

Case-III Let $F_3 = F_3(P_i, Q_i, t)$

Now $F_3(P_i, Q_i, t)$ can be ~~written~~ obtained from $F_1(q_i, Q_i, t)$ by using Legendre's transformation

$$F_1(q_i, Q_i, t) \xrightarrow{L.T} F_3(P_i, Q_i, t)$$

$f(x,y) \xrightarrow{L.T} g(u,y)$

$g(u,y) = f(x,y) - \sum u_i x_i$

$u = \frac{\partial f}{\partial x}$

$p_i = \frac{\partial F_1}{\partial q_i}$

$$F_3(p_i, Q_i, t) = F_1(q_i, Q_i, t) - \sum p_i q_i$$

$$\Rightarrow F_1(q_i, Q_i, t) = F_3(p_i, Q_i, t) + \sum p_i q_i$$

As $\frac{dF_1}{dt} = \sum p_i \dot{q}_i - H - (\sum P_i \dot{Q}_i - H')$

$$\Rightarrow \sum p_i \dot{q}_i - H - \sum P_i \dot{Q}_i + H' = \frac{d}{dt}\left(F_3(p_i, Q_i, t) + \sum p_i q_i\right)$$

$$= \frac{d}{dt} F_3(p_i, Q_i, t) + \frac{d}{dt}\sum p_i q_i$$

$$= \sum \frac{\partial F_3}{\partial p_i} \dot{p} + \sum \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t} + \sum p_i \dot{q}_i + \sum \dot{p}_i q_i$$

$$\Rightarrow \sum \frac{\partial F_3}{\partial p_i} \dot{p} + \sum \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t} = -\sum q_i \dot{p}_i - \sum P_i \dot{Q}_i + H' - H$$

Comparing both sides

$$q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}, \quad H' - H = \frac{\partial F_3}{\partial t}$$