Example – VIII

1 (i) $\lim\limits_{x\to 0} \dfrac{x-\sin x\cos x}{x^3}$.

$\lim\limits_{x\to 0} \dfrac{x-\sin x\cos x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$.

$= \lim\limits_{x\to 0} \dfrac{1-\cos^2 x+\sin^2 x}{3x^2}$

$= \lim\limits_{x\to 0} \dfrac{1-\cos 2x}{3x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{2\sin 2x}{6x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{4\cos 2x}{6}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$

(viii) $\lim\limits_{x\to 0} \dfrac{e^x+\sin x-1}{\log(1+x)}$.

$\lim\limits_{x\to 0} \dfrac{e^x+\sin x-1}{\log(1+x)}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x+\cos x}{\frac{1}{1+x}}$

$= \dfrac{1+1}{\frac{1}{1+0}} = 2.$

(xiii) $\lim\limits_{x\to 0} \dfrac{2\sin x-\sin 2x}{\tan^3 x}$.

$\lim\limits_{x\to 0} \dfrac{2\sin x-\sin 2x}{\tan^3 x}$

$= \lim\limits_{x\to 0} \dfrac{2\sin x-2\sin x\cos x}{\frac{\sin^3 x}{\cos^3 x}}$

$= \lim\limits_{x\to 0} \dfrac{2\sin x(1-\cos x)\cos^3 x}{\sin^3 x}$

$= \lim\limits_{x\to 0} \dfrac{2(\cos^3 x-\cos^4 x)}{\sin^2 x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{2(-3\cos^2 x\sin x+4\cos^3 x\sin x)}{2\sin x\cos x}$

$= \lim\limits_{x\to 0} (-3\cos x+4\cos^2 x)$

$= -3+4$

$= 1$

(xvi) $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-2\cos x}{x\sin x}$

$\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-2\cos x}{x\sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x-e^{-x}+2\sin x}{\sin x+x\cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x+e^{-x}+2\cos x}{\cos x+\cos x-x\sin x}$

$= \dfrac{1+1+2}{1+1-0} = 2.$

(xi) $\lim\limits_{x\to 0} \dfrac{\tan nx-n\tan x}{n\sin x-\sin nx}$.

$\lim\limits_{x\to 0} \dfrac{\tan nx-n\tan x}{n\sin x-\sin nx}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{n\sec^2 nx-n\sec^2 x}{n\cos x-n\cos nx}$

$= \lim\limits_{x\to 0} \dfrac{\sec^2 nx-\sec^2 x}{\cos x-\cos nx}$

$= \lim\limits_{x\to 0} \dfrac{\frac{1}{\cos^2 nx}-\frac{1}{\cos^2 x}}{\cos x-\cos nx}$

$= \lim\limits_{x\to 0} \dfrac{\cos^2 x-\cos^2 nx}{(\cos x-\cos nx)\cos^2 nx\cos^2 x}$

$= \lim\limits_{x\to 0} \dfrac{\cos x+\cos nx}{\cos^2 nx\cos^2 x}$

$= 2.$

(xiv) $\lim\limits_{x\to 0} \dfrac{\sin\log(1+x)}{\log(1+\sin x)}$

$\lim\limits_{x\to 0} \dfrac{\sin\log(1+x)}{\log(1+\sin x)}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{\cos\log(1+x)\cdot\frac{1}{1+x}}{\frac{1}{1+\sin x}\cdot\cos x}$

$= 1.$

(xvi) $\displaystyle \lim_{x\to\frac{\pi}{2}}\left[\frac{(\frac{\pi}{2}-x)\log\sin x}{e^{\cos x}-1+\log(1+x-\frac{\pi}{2})}+\frac{\cos x}{x-\frac{\pi}{2}}\right]$

Now

$$\lim_{x\to\frac{\pi}{2}}\left[\frac{(\frac{\pi}{2}-x)\log\sin x}{e^{\cos x}-1+\log(1+x-\frac{\pi}{2})}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\log\sin x+(\frac{\pi}{2}-x)\frac{\cos x}{\sin x}}{e^{\cos x}(-\sin x)+\frac{1}{1+x-\frac{\pi}{2}}}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\log\sin x+(\frac{\pi}{2}-x)\cot x}{e^{\cos x}(-\sin x)+(1+x-\frac{\pi}{2})^{-1}}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\frac{\cos x}{\sin x}-\cot x-(\frac{\pi}{2}-x)\operatorname{cosec}^2 x}{e^{\cos x}(\sin^2 x)-\cos x\, e^{\cos x}-(1+x-\frac{\pi}{2})^{-2}}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{-2\cot x-(\frac{\pi}{2}-x)\operatorname{cosec}^2 x}{(\sin^2 x-\cos x)e^{\cos x}-(1+x-\frac{\pi}{2})^{-2}}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{2\operatorname{cosec}^2 x+\operatorname{cosec}^2 x-(\frac{\pi}{2}-x)\cdot 2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)}{-\sin x(\sin^2 x-\cos x)e^{\cos x}+(2\sin x\cos x+\sin x)e^{\cos x}+2(1+x-\frac{\pi}{2})^{-3}}\right]$$

$$=\frac{2+1-0}{(-1)(1-0)\cdot 1+(0+1)e^0+2}$$

$$=\frac{3}{2}.$$

and

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}} \quad \left[\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x}{1}=-1$$

$$\therefore \lim_{x\to\frac{\pi}{2}}\left[\frac{(\frac{\pi}{2}-x)\log\sin x}{e^{\cos x}-1+\log(1+x-\frac{\pi}{2})}+\frac{\cos x}{x-\frac{\pi}{2}}\right]$$

$$=\frac{3}{2}-1$$

$$=\frac{1}{2}.$$

$$\therefore \lim\left[\frac{(\pi/2-x)\log\sin x}{e^{\cos x}-1+\log(1+x-\pi/2)}+\frac{\cos x}{x-\pi/2}\right].$$

$$=\frac{3}{2}+\frac{1}{2}$$

$$=2$$

2(ii) $\lim_{x\to 0} \frac{\log x^2}{\log(\cot^2 x)}$

$\lim_{x\to 0} \frac{\log x^2}{\log(\cot^2 x)}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0} \frac{\frac{2x}{x^2}}{\frac{1}{\cot^2 x} 2\cot x(-\operatorname{cosec}^2 x)}$

$= \lim_{x\to 0} \frac{-\sin x \cos x}{x}$

$= \lim_{x\to 0} \frac{-2\sin 2x}{2x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{-2\cos 2x}{2}$

$= -1.$

2(iii) $\lim_{x\to\infty} \frac{x^n}{e^x}$.

$\lim_{x\to\infty} \frac{x^n}{e^x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to\infty} \frac{n x^{n-1}}{e^x}$ ($\frac{\infty}{\infty}$ form)

$= \cancel{\lim_{x\to\infty}} \cancel{n} \cancel{x^n}$

$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$ ($\frac{\infty}{\infty}$ form)

$= \cdots\cdots$

$= \lim_{x\to\infty} \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{e^x}$

$= \lim_{x\to\infty} \frac{n!}{e^x} = 0.$

$= \cancel{1/4}$

(v) $\lim_{x\to 0+} \frac{\log x - \cot\frac{\pi x}{2}}{\cot \pi x}$.

$\lim_{x\to 0+} \frac{\log x - \cot\frac{\pi x}{2}}{\cot \pi x}$

$= \lim_{x\to 0+} \frac{\log x}{\cot \pi x}$ ($\frac{\infty}{\infty}$ form) $- \lim_{x\to 0+} \frac{\cot\frac{\pi x}{2}}{\cot \pi x}$

$= \lim_{x\to 0+} \frac{1/x}{-\pi \operatorname{cosec}^2 \pi x} - \lim_{x\to 0+} \frac{\cos\frac{\pi}{2}x \sin \pi x}{\sin\frac{\pi x}{2} \cos \pi x}$

$= -\frac{1}{\pi} \lim_{x\to 0+} \frac{\sin^2 \pi x}{x}$ ($\frac{0}{0}$ form) $- \lim_{x\to 0+} \frac{\cos\frac{\pi}{2}x \cdot 2\cancel{\sin\frac{\pi}{2}x} \cos\frac{\pi}{2}x}{\cancel{\sin\frac{\pi}{2}x} \cos \pi x}$

$= -\frac{1}{\pi} \lim_{x\to 0+} \frac{2\pi \sin \pi x \cos \pi x}{1} - \frac{2}{1} = 0 - 2 = -2.$

2 (vi) $\lim_{x\to 0+} \log_{\tan^2 x}(\tan^2 2x)$ (vii) $\lim_{x\to\infty}$

$\lim_{x\to 0+} \log_{\tan^2 x}(\tan^2 2x)$

$= \lim_{x\to 0+} \frac{\log(\tan^2 2x)}{\log \tan^2 x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0+} \frac{2\log(\tan 2x)}{2\log \tan x}$

$= \lim_{x\to 0+} \frac{\cancel{2}\cdot\frac{1}{\tan 2x}\sec^2 2x\cdot 2}{\cancel{2}\cdot\frac{1}{\tan x}\cdot\sec^2 x}$

$= \lim_{x\to 0+} \frac{2\sec^2 2x \tan x}{\sec^2 x \tan 2x}$

$= \lim_{x\to 0+} \frac{\sec^2 2x}{\sec^2 x}\cdot\frac{\frac{\tan x}{x}}{\frac{\tan 2x}{2x}}$

$= 1\cdot 1 = 1.$

3 (ii) $\lim_{x\to 1} \operatorname{cosec}(\pi x)\log x$.

$\lim_{x\to 1} \operatorname{cosec}(\pi x)\log x$ ($\infty\times 0$ form)

$= \lim_{x\to 1} \frac{\log x}{\sin(\pi x)}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 1} \frac{1/x}{\pi\cos\pi x}$

$= -\frac{1}{\pi}$

(iv) $\lim_{x\to\frac{\pi}{2}} \sec x(x\sin x - \frac{\pi}{2})$.

$\lim_{x\to\frac{\pi}{2}} \sec x(x\sin x - \frac{\pi}{2})$ ($\infty\times 0$ form)

$= \lim_{x\to\frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to\frac{\pi}{2}} \frac{x\cos x + \sin x}{-\sin x}$

$= -1.$

3. (iii) $\lim_{x\to 0} x\log\sin^2 x$

$\lim_{x\to 0} x\log\sin^2 x$ ($0\times\infty$ form)

$= \lim_{x\to 0} \frac{\log\sin^2 x}{1/x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0} \frac{\frac{1}{\sin^2 x}\cdot 2\sin x\cos x}{-1/x^2}$

$= \lim_{x\to 0} \frac{2x^2}{-\tan x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{4x}{-\sec^2 x} = 0$

(v) $\lim_{x\to 0} \sin x \log x^2$

$\lim_{x\to 0} \sin x\log x^2$ ($0\times\infty$ form)

$= \lim_{x\to 0} \frac{\log x^2}{\operatorname{cosec} x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0} \frac{\frac{1}{x^2}\cdot 2x}{-\operatorname{cosec} x\cot x}$

$= \lim_{x\to 0} \frac{-2\sin x\tan x}{x}$

$= \lim_{x\to 0} \frac{-2\sin^2 x}{x\cos x}$

$= \lim_{x\to 0} \frac{-(1-\cos 2x)}{x\cos x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{-2\sin 2x}{\cos x - x\sin x}$

$= 0.$

(viii) $\lim_{x\to 0+} x^m(\log x)^n$, m, n being positive.

$\lim_{x\to 0+} x^m(\log x)^n$, ($0\times\infty$ form)

First suppose n is positive integer, then

$\lim_{x\to 0+} \frac{(\log x)^n}{\frac{1}{x^m}}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0+} \frac{n(\log x)^{n-1}\cdot\frac{1}{x}}{-m\cdot\frac{1}{x^{m+1}}}$

$= \lim_{x\to 0+} \frac{n(\log x)^{n-1}}{-m\cdot\frac{1}{x^m}}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0+} \dfrac{n(n-1)(\log x)^{n-2}}{(-m)^2 \frac{1}{x^m}}$ ($\frac{\infty}{\infty}$ form)

$\vdots$

$= \lim_{x\to 0+} \dfrac{n(n-1)(n-2)\cdots(\log x)^{n-n}}{(-m)^n x^{-m}}$

$= \dfrac{\lfloor n}{(-m)^n} \lim_{x\to 0+} x^m$

$= \dfrac{\lfloor n}{(-m)^n} \times 0$

$= 0$

Alternative process (49)

$\lim_{x\to 0+} x^m(\log x)^n = \lim_{x\to 0+}\left(\dfrac{\log x}{x^{-\frac{m}{n}}}\right)^n$

$= \lim_{x\to 0+}\left[x^{\frac{m}{n}}(\log x)\right]^n$ ($0\times\infty$ form)

$= \lim_{x\to 0+}\left[\dfrac{\log x}{x^{-\frac{m}{n}}}\right]^n$.

Now, $\lim_{x\to 0+} \dfrac{\log x}{x^{-\frac{m}{n}}}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to 0+} \dfrac{\frac{1}{x}}{(-\frac{m}{n})x^{-\frac{m}{n}-1}}$

$= \lim_{x\to 0+} (-\frac{n}{m}) x^{\frac{m}{n}}$

$= 0$. [$\because$ m & n are positive]

$\therefore \lim_{x\to 0+} x^m(\log x)^n = 0^n = 0$ as n is positive and finite.

Let m>0 be arbitrary. Take a positive integer p>n. Then

$\lim_{x\to 0+} x^m(\log x)^n$

$= \lim_{x\to 0+} x^m(\log x)^p \cdot \dfrac{1}{(\log x)^{p-n}}$

$= \lim_{x\to 0+} x^m(\log x)^p \cdot \lim_{x\to 0+} \dfrac{1}{(\log x)^{p-n}}$

$= 0\times 0$

$= 0$.

(iii) $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$

$\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$ ($\infty-\infty$ form)

$= \lim_{x\to 0} \dfrac{\sin^2 x - x^2}{x^2\sin^2 x}$

$= \lim_{x\to 0} \dfrac{2\sin^2 x - 2x^2}{2x^2\sin^2 x}$

$= \lim_{x\to 0} \dfrac{1-\cos 2x - 2x^2}{x^2(1-\cos 2x)}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \dfrac{2\sin 2x - 4x}{2x(1-\cos 2x)+x^2\cdot 2\sin 2x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \dfrac{4\cos 2x - 4}{2(1-\cos 2x)+4x\sin 2x+4x\sin 2x+4x^2\cos 2x}$

$= \lim_{x\to 0} \dfrac{4\cos 2x - 4}{2(1-\cos 2x)+8x\sin 2x+4x^2\cos 2x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \dfrac{-8\sin 2x}{4\sin 2x+8\sin 2x+16x\cos 2x+8x\cos 2x+8x^2\sin 2x}$

$= \lim_{x\to 0} \dfrac{-2\sin 2x}{3\sin 2x+16x\cos 2x+8x\cos 2x-8x^2\sin 2x}$ ($\frac{0}{0}$) form

$= \lim_{x\to 0} \dfrac{-4\cos 2x}{6\cos 2x+16\cos 2x-32x\sin 2x+8x\cos 2x-16x\sin 2x-16x\sin 2x+16x^2\cos 2x}$

$$= \lim_{x \to 0} \frac{-2\sin 2x}{3\sin 2x + 6x\cos 2x + \cancel{x\cos 2x} - 2x^2\sin 2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-4\cos 2x}{6\cos 2x + 6\cos 2x - 12x\sin 2x - 4x\sin 2x - 4x^2\cos 2x}$$

$$= \frac{-4}{6+6} = -\frac{1}{3}$$

(v) $\lim_{x \to 2} \left(\frac{4}{x^2-4} - \frac{1}{x-2}\right)$

$\lim_{x \to 2} \left(\frac{4}{x^2-4} - \frac{1}{x-2}\right)$ ($\infty - \infty$ form)

$$= \lim_{x \to 2} \left\{\frac{4-(x+2)}{(x-2)(x+2)}\right\}$$

$$= \lim_{x \to 2} \frac{-(x-2)}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{-1}{x+2} = -\frac{1}{4}$$

(iv) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$

$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ ($\infty - \infty$)

$$= \lim_{x \to 1} \left\{\frac{x\log x - x + 1}{(x-1)\log x}\right\} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{\log x}{\log x + \frac{x-1}{x}}$$

$$= \lim_{x \to 1} \frac{\log x}{\log x + 1 - x^{-1}} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{1/x}{1/x + x^2}$$

$$= \frac{1}{2}.$$

(vi) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$.

$\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$ ($\infty - \infty$ form)

$$= \lim_{x \to 0} \left[\frac{x - \log(1+x)}{x^2}\right] \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left(\frac{1 - \frac{1}{1+x}}{2x}\right)$$

$$= \lim_{x \to 0} \left(\frac{x/(x+1)}{2x}\right)$$

$$= \lim_{x \to 0} \left[\frac{1}{2(x+1)}\right] = \frac{1}{2}$$

(viii) $\lim_{x \to \infty} \left(\sqrt{x^2+2x} - x\right)$

Putting $x = \frac{1}{z}$. Since $x \to \infty$, $z \to 0$.

So $\lim_{x \to \infty} \left(\sqrt{x^2+2x} - x\right)$ becomes

$$\lim_{z \to 0} \left(\sqrt{\frac{1}{z^2} + \frac{2}{z}} - \frac{1}{z}\right)$$

$$= \lim_{z \to 0} \frac{\sqrt{1+2z} - 1}{z} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{z \to 0} \frac{\frac{1}{2} \cdot \frac{1}{\sqrt{1+2z}} \cdot 2}{1}$$

$$= \lim_{z \to 0} \frac{1}{\sqrt{1+2z}}$$

$$= 1.$$

5 (iv) $\lim_{x\to 0} (\cos x)^{\cot^2 x}$.

Let $y = (\cos x)^{\cot^2 x}$.

$\therefore \log y = \cot^2 x \log(\cos x)$.

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \cot^2 x \log(\cos x)$ ($\infty \times 0$ form)

$= \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{-\tan x}{2\tan x \sec^2 x}$

$= \lim_{x\to 0} \frac{-1}{2\sec^2 x}$

or, $\lim_{x\to 0} \log y = -\frac{1}{2}$

or, $\log \lim_{x\to 0} y = -\frac{1}{2}$

or $\lim_{x\to 0} y = e^{-1/2}$

or, $\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$

(v) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$.

Let $y = (\sin x)^{\tan x}$

$\therefore \log y = \tan x \log \sin x$

$\therefore \lim_{x\to \frac{\pi}{2}} \log y = \lim_{x\to \frac{\pi}{2}} \tan x \log \sin x$ ($\infty \times 0$ form)

$= \lim_{x\to \frac{\pi}{2}} \frac{\log \sin x}{\cot x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{-\operatorname{cosec}^2 x}$

$= \lim_{x\to \frac{\pi}{2}} (-\cos x \sin x)$

or $\lim_{x\to \frac{\pi}{2}} \log y = 0$

or, $\log \lim_{x\to \frac{\pi}{2}} y = 0$

or, $\lim_{x\to \frac{\pi}{2}} y = 1$

or, $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$.

5.

(vi) $\underset{x\to 1}{Lt}\ x^{\frac{1}{1-x}}$

Let $y = x^{\frac{1}{1-x}}$

$\therefore \log y = \frac{1}{1-x}\log x$

$\therefore \underset{x\to 1}{Lt}\log y = \underset{x\to 1}{Lt}\frac{\log x}{1-x}$ ($\frac{0}{0}$ form)

$= \underset{x\to 1}{Lt}\frac{1/x}{-1}$

$= -1$

or, $\log \underset{x\to 1}{Lt}\ y = -1$

or, $\underset{x\to 1}{Lt}\ x^{\frac{1}{1-x}} = e^{-1}$

(ix) $\underset{x\to 1-}{Lt}\ (1-x^2)^{\frac{1}{\log(1-x)}}$

Let $y = (1-x^2)^{\frac{1}{\log(1-x)}}$

$\therefore \log y = \frac{\log(1-x^2)}{\log(1-x)}$

$\therefore \underset{x\to 1-}{Lt}\log y = \underset{x\to 1-}{Lt}\frac{\log(1-x^2)}{\log(1-x)}$ ($\frac{\infty}{\infty}$ form)

$= \underset{x\to 1-}{Lt}\frac{-\frac{2x}{1-x^2}}{-\frac{1}{1-x}}$

$= \underset{x\to 1-}{Lt}\frac{2x(1-x)}{1-x^2}$

$= \underset{x\to 1-}{Lt}\frac{2x}{1+x} = 1$

or, $\log \underset{x\to 1-}{Lt}\ y = 1$

or, $\underset{x\to 1-}{Lt}\ y = e$

or, $\underset{x\to 1-}{Lt}(1-x^2)^{\frac{1}{\log(1-x)}} = e.$

(xiii) $\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^{1/x}$

Let $y = \left(\frac{\sin x}{x}\right)^{1/x}$

$\therefore \log y = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$.

$\therefore \underset{x\to 0}{Lt}\log y = \underset{x\to 0}{Lt}\frac{\log\left(\frac{\sin x}{x}\right)}{x}$ ($\frac{0}{0}$ form), since $\frac{\sin x}{x}\to 1$ as $x\to 0$

$= \underset{x\to 0}{Lt}\ \frac{x}{\sin x}\cdot$

$= \underset{x\to 0}{Lt}\ \frac{x}{\sin x}\cdot\frac{\sin x - x\cos x}{\cancel{\sin^2 x}}$

(viii) $\underset{x\to 0}{Lt}\left(\frac{1}{x^2}\right)^{\tan x}$.

Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

$\therefore \log y = \tan x \log x^{-2}$

$\therefore \underset{x\to 0}{Lt}\log y = \underset{x\to 0}{Lt}\tan x \log x^{-2}$

$= \underset{x\to 0}{Lt}\frac{\log x^{-2}}{\cot x}$ ($\frac{\infty}{\infty}$ form)

$= \underset{x\to 0}{Lt}\frac{\frac{1}{x^{-2}}\cdot(-2x^{-3})}{-\operatorname{cosec}^2 x}$.

$= \underset{x\to 0}{Lt}\frac{2\sin^2 x}{x}$ ($\frac{0}{0}$ form)

$= \underset{x\to 0}{Lt}\frac{1-\cos 2x}{x}$ ($\frac{0}{0}$ form)

$= \underset{x\to 0}{Lt}(2\sin 2x)$

$= 0.$

or $\underset{x\to 0}{Lt}\log y = 0$

or, $\log \underset{x\to 0}{Lt}\ y = 0$

or, $\underset{x\to 0}{Lt}\ y = 1$

or, $\underset{x\to 0}{Lt}\left(\frac{1}{x^2}\right)^{\tan x} = 1$

(xi) $\underset{x\to\infty}{Lt}\left(1+\frac{1}{x^2}\right)^x$.

Let $y = \left(1+\frac{1}{x^2}\right)^x$.

$\therefore \log y = x\log\left(1+\frac{1}{x^2}\right)$

$\therefore \underset{x\to\infty}{Lt}\log y = \underset{x\to\infty}{Lt}\ x\log\left(1+\frac{1}{x^2}\right)$ ($\infty\times 0$ form)

$= \underset{x\to\infty}{Lt}\frac{\log(1+x^{-2})}{1/x}$ ($\frac{0}{0}$ form)

$= \underset{x\to\infty}{Lt}\frac{\frac{1}{1+x^{-2}}(-2x^{-3})}{-1/x^2}$

$= \underset{x\to\infty}{Lt}\frac{2x}{1+x^2}$ ($\frac{\infty}{\infty}$ form)

$= \underset{x\to\infty}{Lt}\frac{2}{2x}$.

$= 0.$

$\therefore \underset{x\to\infty}{Lt}\left(1+\frac{1}{x^2}\right)^x = 1$

$$= \lim_{x\to 0} \frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{1}$$

$$= \lim_{x\to 0} \frac{x\cos x-\sin x}{x\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$$

$$= 0.$$

or, $\log \lim_{x\to 0} y = 0$

or, $\lim_{x\to 0} y = 1.$

or, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1.$

(xiii) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

Let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$

$\therefore \log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$, since $\lim_{x\to 0}\frac{\tan x}{x} = 1$

$\therefore \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x}\cdot\frac{x\sec^2 x-\tan x}{x^2}}{2x}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{2x^2\tan x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cancel{\sec^2 x} + 2x\sec^2 x\tan x - \cancel{\sec^2 x}}{4x\tan x + 2x^2\sec^2 x}$$

$$= \lim_{x\to 0} \frac{\sec^2 x\tan x}{2\tan x + x\sec^2 x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sec^4 x + 2\tan^2 x\sec^2 x}{2\sec^2 x + \sec^2 x + 2x\sec^2 x\tan x}$$

$$= \frac{1}{3}$$

or, $\log \lim_{x\to 0} y = \frac{1}{3}$

or, $\lim_{x\to 0} y = e^{1/3}$

or, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$

5 (xiv) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Let $y = \left(\frac{\sin x}{x}\right)^{1/x^2}$

$\therefore \log y = \frac{1}{x^2}\log\frac{\sin x}{x}$

$\therefore \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$ $\left(\frac{0}{0}\text{ form}\right)$, since $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$$= \lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2\cos x}$$

$$= \lim_{x\to 0}\frac{-x\sin x}{4x\sin x + 2x^2\cos x}$$

$$= \lim_{x\to 0}\frac{-\sin x}{4\sin x + 2x\cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-\cos x}{4\cos x + 2\cos x - 2x\sin x}$$

$$= -\frac{1}{6}$$

or, $\log \lim_{x\to 0} y = -\frac{1}{6}$

or, $\lim_{x\to 0} y = e^{-1/6}$

or $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$

10. If $\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a, and the limit.

$$\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3\tan^2 x\sec^2 x}.$$

Since denominator is zero as $x \to 0$, so the value of the above limit may be ~~finite~~ finite if the numerator is zero as $x \to 0$

i.e. if $a - 2 = 0$

i.e. if $a = 2$.

When $a = 2$, the given limit become

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{\tan^3 x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\cos x - 2\cos 2x}{3\tan^2 x \sec^2 x}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{-\sin x + 2\sin 2x}{2\tan x \sec^4 x + 2\sec^2 x \tan^3 x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{-\cos x + 4\cos 2x}{2\sec^6 x + 8\sec^4 x \tan^2 x + 4\sec^2 x \tan^4 x + 6\tan^2 x \sec^4 x}$$

$$= \frac{2}{3} \cdot \frac{-1 + 4}{2 + 0 + 0 + 0}$$

$$= 1$$

11. Adjust the constants a and b in order that

$$\lim_{\theta\to 0} \frac{\theta(1+a\cos\theta) - b\sin\theta}{\theta^3} = 1.$$

Now $\lim_{\theta\to 0} \frac{\theta(1+a\cos\theta)-b\sin\theta}{\theta^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{\theta\to 0} \frac{(1+a\cos\theta) - a\theta\sin\theta - b\cos\theta}{3\theta^2}$$

Since the denominator is 0 as $\theta\to 0$, so the value of the above limit may be finite if the numerator is zero as $\theta\to 0$.

$\therefore$ $1+a-b=0$

ie $b = 1+a.$

So the above limits becomes

$$\lim_{\theta\to 0} \frac{1+a\cos\theta - a\theta\sin\theta - (1+a)\cos\theta}{3\theta^2}$$

$$= \lim_{\theta\to 0} \frac{1 - a\theta\sin\theta - \cos\theta}{3\theta^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{\theta\to 0} \frac{-a\sin\theta - a\theta\cos\theta + \sin\theta}{6\theta}$$

$$= \lim_{\theta\to 0} \frac{-(a-1)\sin\theta - a\theta\cos\theta}{6\theta} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{\theta\to 0} \frac{-(a-1)\cos\theta - a\cos\theta + a\theta\sin\theta}{6}$$

$$= \frac{-(a-1)-a}{6}$$

$$= \frac{-2a+1}{6}.$$

$\therefore$ $\frac{-2a+1}{6} = 1$

or, $-2a = 5$

or, $a = -5/2$.

Hence $b = 1-5/2 = -3/2$

12. Determine the value of a, b, c so that

(i) $\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \to 2$, as $x\to 0$.

Since the denominator is zero as $x\to 0$, so the value of of the given limit may be finite if the numerator is zero as $x\to 0$.

$\therefore$ $a-b+c=0$ ---(i)

Now $\underset{x\to 0}{Lt} \dfrac{ae^{x} - b\cos x + ce^{-x}}{x\sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{Lt} \dfrac{ae^{x} + b\sin x - ce^{-x}}{\sin x + x\cos x}$

Since the denominator is zero as $x\to 0$, so the value of the given above limit may be finite if the numerator is zero as $x\to 0$

$\therefore \quad a - c = 0 \quad --- (2)$

Again $\underset{x\to 0}{Lt} \dfrac{ae^{x} + b\sin x + ce^{-x}}{\sin x + x\cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{Lt} \dfrac{ae^{x} + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x}$

$= \dfrac{a+b+c}{2}$.

$\therefore \quad \dfrac{a+b+c}{2} = 2$

ie $a+b+c = 4. \quad --- (3)$

Adding (1) & (3), we get

$2(a+c) = 4$

ie $a+c = 2. \quad --- (4)$

Solving (2) & (4), we get

$a = c = 1$.

Hence from (3)

$b = 2$.

Evaluate the following

13. (i) $\underset{x\to 0}{Lt} \dfrac{xe^{x} - \log(1+x)}{x^2}$

$\underset{x\to 0}{Lt} \dfrac{xe^{x} - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{Lt} \dfrac{(x+1)e^{x} - \frac{1}{x+1}}{2x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{Lt} \dfrac{(x+2)e^{x} + \frac{1}{(x+1)^2}}{2}$

$= \dfrac{3}{2}$.

(iii) $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$

$\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{-\sin x - \sin x - x\cos x + \frac{1}{(1+x)^2}}{2}$

$= \frac{1}{2}$

14. $\lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}$

$\lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x - 4x}{x^5}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x - 4}{5x^4}$

$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\sin x}{20x^3}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{60x^2}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{120x}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{120}$

$= \frac{4}{120} = \frac{1}{30}$

21. $\lim_{x\to\infty} \{x - x^2\log(1+\frac{1}{x})\} = \frac{1}{2}$

$\lim_{x\to\infty} \{x - x^2\log(1+x^{-1})\}$

Put $x = \frac{1}{z}$. As $x\to\infty$, $z\to 0$

So, the above limit becomes

$\lim_{z\to 0} \{\frac{1}{z} - \frac{1}{z^2}\log(1+z)\}$ ($\infty - \infty$ form)

$= \lim_{z\to 0} \frac{z - \log(1+z)}{z^2}$ ($\frac{0}{0}$ form)

$= \lim_{z\to 0} \frac{1 - \frac{1}{1+z}}{2z}$

$= \lim_{z\to 0} \frac{z}{2z(z+1)}$

$= \lim_{z\to 0} \frac{1}{2(z+1)} = \frac{1}{2}$.

1 (xv) Now $\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$

$= 2\sin x\cos x + 2(1-\cos^2 x) - 2\sin x$

$= 2\sin x(\cos x - 1) + 2(1-\cos^2 x) = 2(1-\cos x)(1+\cos x - \sin x)$

and $\cos x - \cos^2 x = \cos x(1-\cos x)$.

$\therefore \lim_{x\to 0}\left(\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}\right)^2$

$= \lim_{x\to 0}\left[\dfrac{2(1+\cos x - \sin x)}{\cos x}\right]^2 = \left\{\dfrac{2(1+1)}{1}\right\}^2 = 16.$

and $\lim_{x\to 0}\dfrac{1-\cos x}{\cos x\sin^2 x} = \lim_{x\to 0}\dfrac{1-\cos x}{\cos x(1-\cos^2 x)}$

$= \lim_{x\to 0}\dfrac{1}{\cos x(1+\cos x)} = \dfrac{1}{2}$

$\therefore \lim_{x\to 0}\left[\left(\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}\right)^2 + \dfrac{1-\cos x}{\cos x\sin^2 x}\right]$

$= 16 + \dfrac{1}{2}$

$= 16\frac{1}{2}$

$$= \underset{x\to 0}{Lt} \frac{-(\sin x / \cos x)}{2\tan x \sec^2 x} = \underset{x\to 0}{Lt}\left(-\tfrac{1}{2}\cos^2 x\right)$$

$$= -\tfrac{1}{2} \left(\because \underset{x\to 0}{Lt} \cos^2 x = 1 \right).$$

Since $\underset{x\to 0}{Lt} \log y = \log \underset{x\to 0}{Lt}\, y$, $\quad \therefore \ \log \underset{x\to 0}{Lt}\, y = -\tfrac{1}{2}$.

$\therefore \ \underset{x\to 0}{Lt}\, y = e^{-\frac{1}{2}}$. $\quad \therefore$ the required limit $= e^{-\frac{1}{2}}$.

Ex. 7. *Show that* $\underset{x\to 0}{Lt} \dfrac{x - \sin x}{x^3} = \dfrac{1}{6}$. [*C.P. 1932, 1995*]

Writing down the expansion of $\sin x$ in a finite power series, we have

$$x - \sin x = x - \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!}\sin\left(\frac{5\pi}{2} + \theta x\right)\right\}, \quad 0 < \theta < 1$$

$$= \frac{x^3}{3!} - \frac{x^5}{5!}\sin\left(\frac{5\pi}{2} + \theta x\right)$$

$$= x^3 \left\{ \frac{1}{3!} - \frac{x^2}{5!}\sin\left(\frac{5\pi}{2} + \theta x\right)\right\},$$

$$\therefore \frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!}\sin\left(\frac{5\pi}{2} + \theta x\right).$$

$$\therefore \underset{x\to 0}{Lt} \frac{x - \sin x}{x^3} = \underset{x\to 0}{Lt} \left\{ \frac{1}{3!} - \frac{x^2}{5!}\sin\left(\frac{5\pi}{2} + \theta x\right)\right\} = \frac{1}{3!} = \frac{1}{6};$$

since $\dfrac{x^2}{5!}\sin\left(\dfrac{5\pi}{2} + \theta x\right) \to 0$ as $x \to 0$, $\left|\sin\left(\dfrac{5\pi}{2} + \theta x\right)\right|$ being ≤ 1.

Note. This being of the form 0/0 can also be obtained by the method of Art. 11.2.

Ex. 8. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{\sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$

Multiplying both the numerator and denominator by

$$\left(\sqrt{a^2 + ax + x^2} + \sqrt{a^2 - ax + x^2}\right)\left(\sqrt{a + x} + \sqrt{a - x}\right)$$

and simplying, the required limit

$$= \underset{x\to 0}{Lt} \frac{2ax\left(\sqrt{a+x}+\sqrt{a-x}\right)}{2x\left(\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}\right)}$$

$$= \underset{x\to 0}{Lt} \frac{a\left(\sqrt{a+x}+\sqrt{a-x}\right)}{\left(\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}\right)}$$

Now, the limit of the numerator $=a.2\sqrt{a}$ and that of the denominator $=2a$. Therefore the required limit $=\sqrt{a}$.

Note. An algebraical or trigonometrical transformation often enables us to obtain the limiting values without using calculus, as shown above, which case belongs to the form 0/0.

Ex. 9. *If* $\underset{x\to 0}{Lt} \frac{\sin 2x + a\sin x}{x^3}$ *be finite, find the value of 'a' and the limit.* *[C.P. 1931, 1994, 2000, 2006]*

The given limit, being of the form 0/0,

$$= \underset{x\to 0}{Lt} \frac{2\cos 2x + a\cos x}{3x^2} \quad (\text{by } \S\ 11.2)$$

When $x\to 0$, the denominator $3x^2=0$; hence, in order that the limiting value of the expression may be finite, the numerator $(2\cos 2x + a\cos x)$ should be zero, as $x\to 0$. $\therefore\ 2+a=0$, *i.e.*, $a=-2$.

When $a=-2$, the given limit becomes

$$= \underset{x\to 0}{Lt} \frac{\sin 2x - 2\sin x}{x^3} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-4\sin 2x + 2\sin x}{6x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-8\cos 2x + 2\cos x}{6} = -\frac{6}{6} = -1.$$

Ex.10. *Evaluate* $\underset{x\to 0}{Lt}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ *[C. P. 1947, 1994, 1997, V.P. 1999]*

Let $\quad u = \underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}.$

$$\therefore \quad \log u = \frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \log\left(\frac{\tan x}{x}\right)\Big/ x.$$

Since $\underset{x\to 0}{Lt}\dfrac{\tan x}{x} = 1$ $\underset{x\to 0}{Lt}\log u$ is of the form $\dfrac{0}{0}$.

$$\therefore \underset{x\to 0}{Lt}\log u = \underset{x\to 0}{Lt}\log\left(\frac{\tan x}{x}\right)\Big/ x$$

$$= \underset{x\to 0}{Lt}\left(\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{x^2}\right)\Big/ 1 \quad \text{(by § 11.2)}$$

$$= \underset{x\to 0}{Lt}\frac{2x - \sin 2x}{x\sin 2x} = \underset{x\to 0}{Lt}\frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x}$$

$$= \underset{x\to 0}{Lt}\frac{4\sin 2x}{2\cos 2x - 4x\sin 2x} = 0.$$

Since $\underset{x\to 0}{Lt}\log u = \log \underset{x\to 0}{Lt} u$, $\quad \therefore \log \underset{x\to 0}{Lt} u = 0.$

$\therefore \quad \underset{x\to 0}{Lt} u = e^0 = 1$, *i.e.*, the required limit $= 1$.

Otherwise **:** Writing the finite form of the expansion of $\tan x$ by Maclaurin's theorem,

$$\tan x = x + \tfrac{1}{3}x^3\alpha \quad \text{where} \quad \alpha = \sec\theta x\left(1 + 2\tan^2\theta x\right), \quad 0 < \theta < 1.$$

$$\log u = \frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \frac{1}{x}\log\frac{x + \frac{1}{3}x^2\alpha}{x} = \frac{1}{x}\log\left(1 + \tfrac{1}{3}x^2\alpha\right)$$

$$= \frac{1}{\frac{1}{3}x^2\alpha}\log\left(1 + \tfrac{1}{3}x^2\alpha\right)\tfrac{1}{3}x\alpha = \frac{1}{v}\log\left(1 + v\right)\cdot\tfrac{1}{3}x\alpha,$$

where $v = \frac{1}{3}x^2\alpha$.

When $x \to 0, v \to 0$, also $\underset{v\to 0}{Lt}\dfrac{1}{v}\log(1+v) = 1.$

Hence, $\underset{x\to 0}{Lt}\log u = \underset{v\to 0}{Lt}\dfrac{1}{v}\log(1+v)\cdot\underset{x\to 0}{Lt}\left(\tfrac{1}{3}xa\right),$

$\therefore \quad \underset{x\to 0}{Lt}\log u = 0.$ Hence, etc.

Ex.11. *Evaluate* $\underset{x\to 0}{Lt}\dfrac{(e^x-1)\tan^2 x}{x^3}$

Given limit $= \underset{x\to 0}{Lt}\left\{\dfrac{e^x-1}{x}\cdot\left(\dfrac{\tan x}{x}\right)^2\right\}$

$= \underset{x\to 0}{Lt}\dfrac{e^x-1}{x}\times\left(\underset{x\to 0}{Lt}\dfrac{\tan x}{x}\right)^2.$

Now, $\underset{x\to 0}{Lt}\dfrac{e^x-1}{x}\left(\text{being of the form }\dfrac{0}{0}\right) = \underset{x\to 0}{Lt}\dfrac{e^x}{1} = e^0 = 1$

Also, $\underset{x\to 0}{Lt}\dfrac{\tan x}{x} = 1$, $\therefore$ the required limit $= 1\times 1^2 = 1$.

Note. Such forms are sometimes called *Compound Indeterminate forms*. In evaluating limits of such forms, the use of the theorems on limit (*Art. 3.8*) is of great help.

11.9. Miscellaneous Worked Out Examples.

Ex. 1. Evaluate : $\lim\limits_{x\to 1}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$ [*C. P. 1981*]

Solution : $\lim\limits_{x\to 1}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$

$= \lim\limits_{x\to 1}\left\{\dfrac{x\log x - x + 1}{(x-1)\log x}\right\}$ [Form $\dfrac{0}{0}$]

$= \lim\limits_{x\to 1}\left\{\dfrac{\log x + x\cdot\dfrac{1}{x} - 1}{\log x + (x-1)\dfrac{1}{x}}\right\}$ [By L'Hospitals Rule]

$= \lim\limits_{x\to 1}\left\{\dfrac{x\log x}{x\log x + x - 1}\right\}$ [Form $\dfrac{0}{0}$]

$= \lim\limits_{x\to 1}\dfrac{1+\log x}{2+\log x} = \dfrac{\lim\limits_{x\to 1}(1+\log x)}{\lim\limits_{x\to 1}(2+\log x)} = \dfrac{1+0}{2+0} = \dfrac{1}{2}.$

Ex. 2. Evaluate : $\lim\limits_{x\to 0}\left\{\frac{1}{x}-\frac{2}{x\left(e^x+1\right)}\right\}$ [*C. P. 1982*]

Solution : $\lim\limits_{x\to 0}\left\{\frac{1}{x}-\frac{2}{x\left(e^x+1\right)}\right\}$

$$=\lim_{x\to 0}\left\{\frac{e^x-1}{x\left(e^x+1\right)}\right\} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{e^x}{1+(x+1)\,e^x}$$

$$=\frac{\lim\limits_{x\to 0} e^x}{\lim\limits_{x\to 0}\left\{1+(x+1)e^x\right\}}=\frac{1}{1+1}=\frac{1}{2}.$$

Ex. 3. Evaluate : $\lim\limits_{x\to 0}\frac{\sin x-x}{x^3}$ [*C. P. 1995*]

Solution : $\lim\limits_{x\to 0}\frac{\sin x-x}{x^3}$ $\left[\text{ Form } \frac{0}{0}\right]$

$$=\lim_{x\to 0}\frac{\cos x-1}{3x^2}$$

$$=\lim_{x\to 0}\frac{-\sin x}{6x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}.$$

Ex. 4. Find the value of

(i) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$. [*C. P. 1989, 98*]

(ii) $\lim\limits_{x\to 1}\left(x^{\frac{1}{1-x}}\right)$. [*C. P. 1995*]

(iii) $\lim\limits_{x\to\infty} (1+x)^{\frac{1}{x}}$. [C. P. 1996]

(iv) $\lim\limits_{x\to 0} \left(x^{2\sin x}\right)$. [C. P. 1996]

(v) $\lim\limits_{x\to 0} (1+\sin x)^{\cot x}$. [C. P. 1993]

Solution : (i) Let $y=(\cos x)^{\frac{1}{x^2}}$

$\therefore \quad \log y = \frac{1}{x^2}\cdot \log\cos x = \frac{\log\cos x}{x^2}$ [Base of logarithm is e]

$\therefore \quad \lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0}\left\{\frac{\log\cos x}{x^2}\right\}$ [Form $\frac{0}{0}$]

$= \lim\limits_{x\to 0} \frac{-\tan x}{2x}$ [Form $\frac{0}{0}$]

$= \lim\limits_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$.

or, $\log\left\{\lim\limits_{x\to 0} y\right\} = -\frac{1}{2}$

$\therefore \quad \lim\limits_{x\to 0} y = e^{-\frac{1}{2}}$

(ii) Let $y = x^{\left(\frac{1}{1-x}\right)}$

or, $\log y = \frac{1}{1-x}\cdot\log x = \frac{\log x}{1-x}$

$\therefore \quad \lim\limits_{x\to 1} \{\log y\} = \lim\limits_{x\to 1} \frac{\log x}{1-x}$ [Form $\frac{0}{0}$]

$= \lim\limits_{x\to 1}\left(-\frac{1}{x}\right) = -1$

$\therefore \quad \log\left\{\lim\limits_{x\to 1} y\right\} = -1$

$\therefore \lim_{x\to 1} y = e^{-1}$

$\therefore \lim_{x\to 1} x^{\left(\frac{1}{1-x}\right)} = e^{-1}$

(iii) $\lim_{x\to\infty} (1+x)^{\frac{1}{x}}$

Let , $z=\dfrac{1}{x}$ then $z\to 0$ as $x\to\infty$

Let $y=(1+x)^{\frac{1}{x}}$

then $\log y = z\cdot\log\left(1+\dfrac{1}{z}\right)$

Thus $\lim_{z\to 0} \{\log y\} = \lim_{z\to 0}\left\{z\cdot\log\left(1+\dfrac{1}{z}\right)\right\}$

$$=\lim_{z\to 0}\frac{\log\left(1+\frac{1}{z}\right)}{\frac{1}{z}} \qquad \left[\text{ Form } \frac{\infty}{\infty}\right]$$

$$=\lim_{z\to 0}\frac{\left(\frac{1}{1+\frac{1}{z}}\times\left(-\frac{1}{z^2}\right)\right)}{\left(-\frac{1}{z^2}\right)}$$

$$=\lim_{z\to 0}\frac{z}{1+z} \qquad [\text{ as } z\to 0,\ z\neq 0\ ]$$

$$=0$$

$\therefore \log\left\{\lim_{x\to\infty} y\right\}=0$

i.e., $\lim_{x\to\infty} y = e^0 = 1$

Thus $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = 1.$

(*iv*) Let, $y = x^{2\sin x}$

or, $\log y = 2\sin x \cdot \log x$ [Base of logarithm is e]

or, $\lim\limits_{x\to 0} \{\log y\} = \lim\limits_{x\to 0} \dfrac{2\log x}{\operatorname{cosec} x}$ [Form $\dfrac{0}{0}$]

$$= \lim_{x\to 0} \frac{\frac{2}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{-x\cos x} \quad \text{[Form } \frac{0}{0} \text{]}$$

$$= -2 \lim_{x\to 0} \frac{2\sin x\cos x}{\cos x - x\sin x}$$

$$= -2 \frac{\lim\limits_{x\to 0} \sin 2x}{\lim\limits_{x\to 0} (\cos x - x\sin x)} = -2 \times \frac{0}{1} = 0.$$

or, $\log\left\{\lim\limits_{x\to 0} y\right\} = 0$

i.e., $\lim\limits_{x\to 0} y = e^0 = 1$

i.e., $\lim\limits_{x\to 0} x^{2\sin x} = 1$

(*v*) Let $y = (1+\sin x)^{\cot x}$

or, $\log y = \cot x \log(1+\sin x)$ [Base of logarithm is e]

$$= \frac{\log(1+\sin x)}{\tan x}$$

$\therefore \lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} \dfrac{\log(1+\sin x)}{\tan x}$ [Form $\dfrac{0}{0}$]

$$= \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{\sec^2 x}$$

$$= \lim_{x\to 0} \frac{\cos^3 x}{1+\sin x} = 1$$

$$\therefore \log\left\{\lim_{x\to 0} y\right\} = 1$$

or, $\lim_{x\to 0} y = e^1 = e$

$$\therefore \lim_{x\to 0}(1+\sin x)^{\cot x} = e$$

Ex. 5. Evaluate :

(i) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ *[B. P. 1995, C. P. 1994, '97]*

(ii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ *[C. P. 1990, '98]*

Solution : (i) Let, $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$$\therefore \lim_{x\to 0}\{\log y\} = \lim_{x\to 0}\left\{\frac{1}{x}\log\left(\frac{\tan x}{x}\right)\right\}$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x}$$ [Form $\frac{0}{0}$ $x\to 0$, since as $x\to 0$

$\frac{\tan x}{x}\to 1$, *i.e.*, $\log\left(\frac{\tan x}{x}\right)\to 0$]

$$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{1}$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x\cos x}-\frac{1}{x}\right)$$

$$s= \lim_{x\to 0}\frac{2x-\sin 2x}{x\sin 2x}$$ [Form $\frac{0}{0}$]

$$= 2\cdot\lim_{x\to 0}\frac{1-\cos 2x}{\sin 2x+2x\cos 2x}$$ [Form $\frac{0}{0}$]

$$= 2\lim_{x\to 0} \frac{2\sin 2x}{2\cos 2x + 2\cos 2x - 4x\sin 2x} \quad \left[\text{ Form } \frac{0}{0}\right]$$

$$= 4\cdot\frac{\lim\limits_{x\to 0} \sin 2x}{\lim\limits_{x\to 0}(4\cos 2x - 4x\sin 2x)} = 4\times\frac{0}{4} = 0$$

or, $\log\left\{\lim\limits_{x\to 0} y\right\} = 0$

or, $\lim\limits_{x\to 0} y = e^0 = 1$

$$\therefore \quad \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1$$

(*ii*) Let, $y = \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

or, $\log y = \dfrac{1}{x^2}\cdot\log\dfrac{\sin x}{x} = \dfrac{\log\frac{\sin x}{x}}{x^2}$

$\because \quad \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right) = 1, \quad \lim\limits_{x\to 0} \log\left(\dfrac{\sin x}{x}\right) = 0$

$$\therefore \quad \lim_{x\to 0}(\log y) = \lim_{x\to 0}\frac{\log\frac{\sin x}{x}}{x^2} \quad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\frac{x}{\sin x}\times\frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x\to 0}\cdot\frac{x\cos x - \sin x}{2x^2\sin x} \quad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2\cos x}$$

$$= \lim_{x\to 0}\frac{-\sin x}{2x\cos x + 4\sin x} \quad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x\sin x + 4\cos x} = -\frac{1}{6}$$

$$\therefore \log\left\{\lim_{x\to 0} y\right\} = -\frac{1}{6}$$

$$\text{or, } \lim_{x\to 0} y = e^{-\frac{1}{6}}$$

$$\therefore \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}.$$

Ex. 6. **Evaluate :**

(i) $\lim_{x\to 1} \frac{\log(1-x)}{\cot(\pi x)}$ [*C. P. 1993*]

(ii) $\lim_{x\to 0} (\cos mx)^{n/x^2}$ [*C. P. 2002*]

Solution : (i) $\lim_{x\to 1} \frac{\log(1-x)}{\cot(\pi x)}$ [Form $\frac{\infty}{\infty}$]

$$= \lim_{x\to 1} \frac{-\frac{1}{(1-x)}}{-\pi\operatorname{cosec}^2(\pi x)}$$

$$= \lim_{x\to 1} \frac{\sin^2(\pi x)}{\pi(1-x)} \quad [\text{ Form } \tfrac{0}{0}]$$

$$= \lim_{x\to 1} \frac{\{2\sin(\pi x)\cos(\pi x)\}\pi}{\pi\times(-1)} = \lim_{x\to 1}(-2\sin 2\pi x) = 0.$$

(ii) $\lim_{x\to 0} (\cos mx)^{n/x^2}$

Let $y = (\cos mx)^{\frac{n}{x^2}}$

then $\log y = \frac{n}{x^2}\log(\cos mx)$

$$\lim_{x\to 0}\{\log y\} = n\cdot\lim_{x\to 0}\frac{\log(\cos mx)}{x^2} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= -\frac{mn}{2}\cdot\lim_{x\to 0}\frac{\tan mx}{x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= -\frac{m^2 n}{2}\cdot\lim_{x\to 0}\frac{\sec^2 mx}{1} = -\frac{1}{2}m^2 n$$

$$\therefore \log\left\{\lim_{x\to 0} y\right\} = -\frac{1}{2}m^2 n$$

or, $\lim_{x\to 0} y = e^{-\frac{1}{2}m^2 n}$

$$\therefore \lim_{x\to 0}(\cos mx)^{\frac{n}{x^2}} = e^{-\frac{1}{2}m^2 n}.$$

Ex. 7. (*i*) Find a, b such that $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3} = 1$ [*C. P. 1990*]

(*ii*) If $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$ is finite, find a and the value of the limit.

[*C. P. 1994, 2000*]

Solution : (*i*) Here, $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$ $\left[\text{ Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{1(1+a\cos x)-ax\sin x-b\cos x}{3x^2}$$

$$= \lim_{x\to 0}\frac{1+(a-b)\cos x-ax\sin x}{3x^2} \qquad \ldots \quad (1)$$

For (1) to be of the form $\frac{0}{0}$, $1+(a-b)=0$

i.e., $b = 1+a$ $\ldots$ (2)

So, the given expression $= \lim_{x\to 0}\frac{1-\cos x-ax\sin x}{3x^2}$ $\left[\text{ Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\sin x - a\sin x - ax\cos x}{6x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - a\cos x - a\cos x + ax\sin x}{6}$$

$$= \lim_{x\to 0} \frac{(1-2a)\cos x + ax\sin x}{6} = \frac{1-2a}{6} = 1, \text{ given}$$

$\therefore\ 1-2a = 6$, *i.e.*, $a = -\frac{5}{2}$

From (2), $b = 1 - \frac{5}{2} = -\frac{3}{2}$

Thus, $a = -\frac{5}{2}$, $b = -\frac{3}{2}$.

(*ii*) $$\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

For this limit to be finite, the form should be $\frac{0}{0}$.

i.e., $2+a=0$, or, $a=-2$

$$\therefore \lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8+2}{6} = -1.$$

Hence, $a=-2$ and the value of the limit is -1.