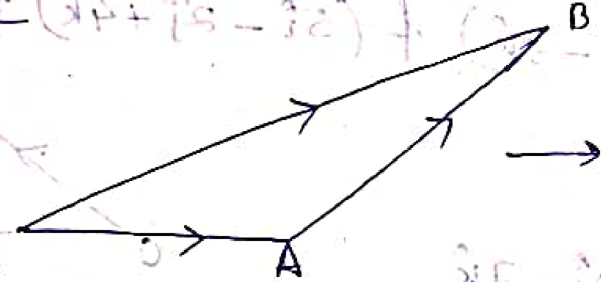
Vector Algebra

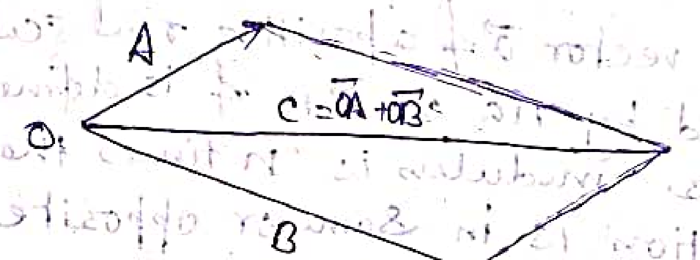
Vector Addition—



→ Triangle law of addition.

The single vector, whose effect is equal to the combined effect of two or more vectors, is called resultant or the sum of these vectors.

$$\underbrace{\overrightarrow{OB}}_{} = \underbrace{\overrightarrow{OA} + \overrightarrow{AB}}_{\text{Components of } \overrightarrow{OB}}$$



→ Parallelogram law.

Addition of more than two vectors—

If $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}$ are five vectors, then resultant vector $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{AB}$

Polygon law of addition of vectors.

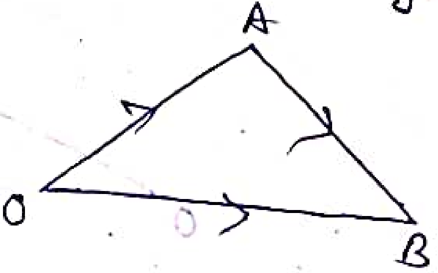
*** Difference of vectors —**

The difference of vectors A & B, denoted by A − B, is that vector C, which added to B, gives A.

A − B may be defined as A + (−B)

If A = B, then A − B is defined as the null or zero vector.

Ex. If the position vectors of A & B, with respect to O, be $(\hat{i}+3\hat{j}-7\hat{k})$ & $(5\hat{i}-2\hat{j}+4\hat{k})$ respectively, then find $\overrightarrow{AB}$.



Soln

Let, $\overrightarrow{OA} = \hat{i}+3\hat{j}-7\hat{k}$

$\overrightarrow{OB} = 5\hat{i}-2\hat{j}+4\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (5\hat{i}-2\hat{j}+4\hat{k}) - (\hat{i}+3\hat{j}-7\hat{k})$

$= 5\hat{i}-2\hat{j}+4\hat{k}-\hat{i}-3\hat{j}+7\hat{k}$

$= 4\hat{i}-5\hat{j}+11\hat{k}$

Scalar Multiplication –

The product of a vector $\vec{a}$ & a positive real scalar quantity n is denoted by $n\vec{a}$ or $\vec{a}.n$ & is defined as the vector whose modulus is n times that of $\vec{a}$ & whose direction is in same or opposite of A according as m is positive or negative.

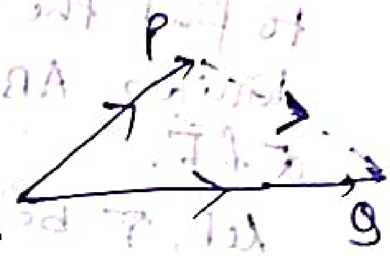
If $n=0$, then $n\vec{a} = 0$, the null vector.

$n \rightarrow$ scalar; $\vec{a} \rightarrow$ vector.

product $\Rightarrow n\vec{a} \Rightarrow$ whose magnitude is n time the magnitude of $\vec{a}$.

Q The position vectors of points P & Q are given by $r_1 = 2\hat{i} + 3\hat{j} - k$, $r_2 = 4\hat{i} - 3\hat{j} + 2\hat{k}$. Determine $\vec{PQ}$ in terms of $\hat{i}$, $\hat{j}$ & $\hat{k}$, find its magnitude.

Solu^n



$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= r_2 - r_1$$

$$= (4\hat{i} - 3\hat{j} + 2\hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k})$$

$$= 2\hat{i} - 6\hat{j} + 3\hat{k}$$

$$|\vec{PQ}| = \sqrt{2^2 + 6^2 + 3^2} \text{ unit}$$

$$= \sqrt{4 + 36 + 9} = \sqrt{49} \text{ unit} = 7 \text{ unit}$$

Q Suppose $A = 3\hat{i} - \hat{j} - 4\hat{k}$, $B = -2\hat{i} + 4\hat{j} - 3\hat{k}$, $C = \hat{i} + 2\hat{j} - \hat{k}$, find i) $2A - B + 3C$,

ii) $|A + B + C|$.

Solu^n

$2A - B + 3C$

$$= 2(3\hat{i} - \hat{j} - 4\hat{k}) - (-2\hat{i} + 4\hat{j} - 3\hat{k}) + 3(\hat{i} + 2\hat{j} - \hat{k})$$

$$= (6\hat{i} - 2\hat{j} - 8\hat{k}) + 2\hat{i} - 4\hat{j} + 3\hat{k} + 3\hat{i} + 6\hat{j} - 3\hat{k}$$

$$= 11\hat{i} - 8\hat{k}$$

ii) $A + B + C$

$$= (3\hat{i} - \hat{j} - 4\hat{k}) + (-2\hat{i} + 4\hat{j} - 3\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})$$

$$= 2\hat{i} + 5\hat{j} - 8\hat{k}$$

Division of a Segment in a given ratio–

Let, the two points A & B have the position vectors $\vec{a}$ & $\vec{b}$ relative to the base point O. It is required to find the position vector of a point R, which divides AB internally in the ratio m:n, in terms of $\vec{a}$ & $\vec{b}$.

Let, $\vec{r}$ be the position vector of R relative to the base point O,

Thus, $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} \text{ if } m+n \neq 0.$$

Note:–

The position vector of the middle point of AB is
$$\frac{1}{2}(\vec{a} + \vec{b})$$

Linear combination of vectors

* A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c} \dots$ if $\exists$ scalars $x, y, z, \dots$, s.t $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \dots$

* A set of vectors $\{\vec{a}, \vec{b}, \vec{c} \dots\}$ is said to be linearly dependent, if $\exists$ a set of scalars $\{x, y, z \dots\}$, not all zero s.t $x\vec{a} + y\vec{b} + z\vec{c} + \dots = 0$.

* A set of vectors $\{\vec{a}, \vec{b}, \vec{c} \dots\}$ is said to be linearly independent if $\exists$ a set of scalars $\{x, y, z, \dots\}$, all zero s.t
$$x\vec{a} + y\vec{b} + z\vec{c} + \dots = 0.$$

Collinear vectors —

Vectors having the same or parallel supports ~~for~~ are called collinear vectors.

The vectors $\vec{a}$ & $m\vec{b}$ are collinear vectors, i.e. $\vec{a} = m\vec{b}$

Coplanar vectors —

A system of vectors having their supports parallel to the same plane is said to be coplanar.

Theorem 1:

If $\vec{a}$ & $\vec{b}$ be two non-zero non-collinear vectors and x, y be two scalars s.t. $x\vec{a} + y\vec{b} = 0$ then $x = 0$ & $y = 0$

Theorem 2

If $\vec{a}$ & $\vec{b}$ be two non-collinear vectors, then every vector $\vec{r}$, coplanar with $\vec{a}$ & $\vec{b}$, can be represented uniquely as a linear combination of $\vec{a}$ & $\vec{b}$.

Theorem 3

If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero non-coplanar vectors and x, y, z be three scalars s.t. $x\vec{a} + y\vec{b} + z\vec{c} = 0$, then $x = y = z = 0$

Theorem 4 —

Any vector can be uniquely represented as a linear combination of three non-coplanar vectors.

Theorem 5

The necessary & sufficient condition for three distinct points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be collinear is that there exist three scalars x, y, z not all zero, s.t.

$$x\vec{a} + y\vec{b} + z\vec{c} = 0, \quad \text{where } x + y + z = 0$$

Q. 1> Show that the following vectors are coplanar:
$3\vec{a}-7\vec{b}-4\vec{c}$, $3\vec{a}-2\vec{b}+\vec{c}$, $\vec{a}+\vec{b}+2\vec{c}$
where $\vec{a}, \vec{b}, \vec{c}$ are any three non-coplanar vectors.

Solⁿ Let us write,

$$3\vec{a}-7\vec{b}-4\vec{c} = x(3\vec{a}-2\vec{b}+\vec{c})+y(\vec{a}+\vec{b}+2\vec{c})$$

where x & y are scalars.

$$\Rightarrow 3\vec{a}-7\vec{b}-4\vec{c} = (3x+y)\vec{a}+(-2x+y)\vec{b}+(x+2y)\vec{c}$$

$\therefore$ $3x+y=3$, $-2x+y=-7$, $x+2y=-4$

Solving these two equations, $x=2$ & $y=-3$.

2> Show that the vectors $(1,2,3)$ & $(4,-2,7)$ are linearly independent.

Solⁿ Let, $\vec{a}=(1,2,3)$, $\vec{b}=(4,-2,7)$

Let, x & y are two scalars.

$$x\vec{a}+y\vec{b}=\vec{0}$$

$$\Rightarrow x(1,2,3)+y(4,-2,7)=\vec{0}$$

$$\Rightarrow (x+4y,\ 2x-2y,\ 3x+7y)=(0,0,0)$$

$\therefore$ $x+4y=0$, $2x-2y=0$, $3x+7y=0$

By solve these equations $x=0$, $y=0$

$\therefore$ They are linearly independent.

Q Show that the three position vectors $(2\hat{i} + 4\hat{j} - \hat{k})$, $(4\hat{i} + 5\hat{j} + \hat{k})$ & $(3\hat{i} + 6\hat{j} - 3\hat{k})$ form a right-angled triangle.

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (4\hat{i} + 5\hat{j} + \hat{k}) - (2\hat{i} + 4\hat{j} - \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (3\hat{i} + 6\hat{j} - 3\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -\hat{i} + \hat{j} - 4\hat{k}$

$\vec{CA} = (2\hat{i} + 4\hat{j} - \hat{k}) - (3\hat{i} + 6\hat{j} - 3\hat{k})$

$= -\hat{i} - 2\hat{j} + 2\hat{k}$

$|\vec{AB}| = \sqrt{4+1+4} = \sqrt{9}$

$|\vec{BC}| = \sqrt{1+1+16} = \sqrt{18}$

$|\vec{CA}| = \sqrt{1+4+4} = \sqrt{9}.$

clearly, $AB^2 + CA^2 = 9 + 9 = 18 = BC^2$

$\therefore$ the points A, B, C form a right-angled triangle.

Q Show that the points $(2,-1,3)$, $(3,-5,1)$ & $(-1,11,9)$ are collinear.

Soln $\vec{OA} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{OB} = 3\hat{i} - 5\hat{j} + \hat{k}$ &

$\vec{OC} = -\hat{i} + 11\hat{j} + 9\hat{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = (3\hat{i} - 5\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$= \hat{i} - 4\hat{j} - 2\hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = (-\hat{i} + 11\hat{j} + 9\hat{k}) - (3\hat{i} - 5\hat{j} + \hat{k})$

$= -4\hat{i} + 16\hat{j} + 8\hat{k}$

Since, $\vec{BC} = -4\vec{AB}$

the point B is common to both the vectors $\vec{AB}$ & $\vec{BC}$, it follows that $\vec{AB}$ & $\vec{BC}$ are collinear i.e the points A, B & C are collinear.

Q Determine the values of λ & μ, for which the vectors $(-3\hat{i} + 4\hat{j} + \lambda\hat{k})$ & $(\mu\hat{i} + 8\hat{j} + 6\hat{k})$ are Collinear.

Solⁿ $(-3\hat{i} + 4\hat{j} + \lambda\hat{k}) = n(\mu\hat{i} + 8\hat{j} + 6\hat{k})$

$\therefore -3 = \mu n, \quad 4 = n8 \quad\quad \lambda = 6n.$

$\Rightarrow -3 = \mu \cdot \frac{1}{2} \quad \Rightarrow n = \frac{1}{2} \quad\quad \Rightarrow \lambda = 6 \times \frac{1}{2} = 3$

$\Rightarrow \mu = -6$

Q If $\vec{a}$ & $\vec{b}$ are non-collinear vectors & $\vec{p} = (x + 4y)\vec{a} + (2x + y + 1)\vec{b}$ & $\vec{q} = (-2x + y + 2)\vec{a} + (2x - 3y - 1)\vec{b}$, then find x & y, so that $3\vec{p} = 2\vec{q}$.

Solⁿ By question, $3\vec{p} = 2\vec{q}$

$\Rightarrow 3[(x + 4y)\vec{a} + (2x + y + 1)\vec{b}] = 2[(-2x + y + 2)\vec{a} + (2x - 3y - 1)\vec{b}] \quad (1)$

Since $\vec{a}$ & $\vec{b}$ are non-collinear vectors, hence coefficient of $\vec{a}$ & $\vec{b}$ on two sides of (1) are equal.

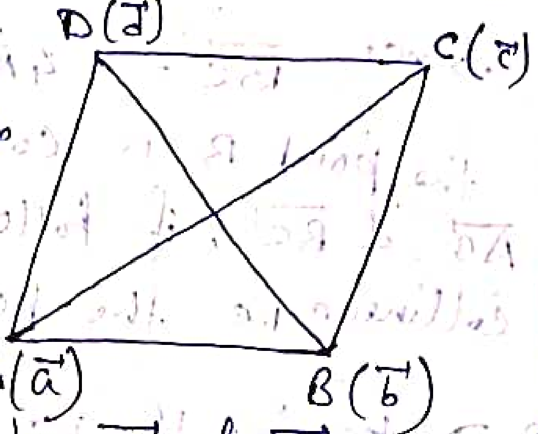
$3(x + 4y) = 2(-2x + y + 2) \Rightarrow 7x + 10y = 4 \quad (2)$

$3(2x + y + 1) = 2(2x - 3y - 1) \Rightarrow 2x + 9y = -5 \quad (3)$

Solving, $y = -1$ & $x = 2$.

Example :

Using vector method show that the quadrillateral whose diagonals bisect each other is a parallelogram.



Soln. If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are the position vectors of the vertices A, B, C & D respectively of the quadrilateral ABCD w.r.t an arbitrary origin, then the position vectors of the mid-points of the diagonals $\overrightarrow{AC}$ & $\overrightarrow{BD}$ are $\frac{\vec{a}+\vec{c}}{2}$ & $\frac{\vec{b}+\vec{d}}{2}$ respectively.

By problem, we have,

$$\frac{\vec{a}+\vec{c}}{2} = \frac{\vec{b}+\vec{d}}{2}$$

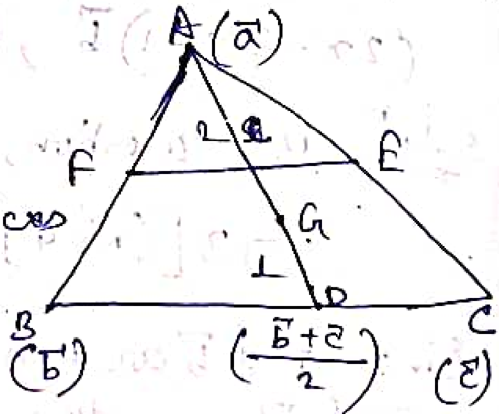
$$\Rightarrow \vec{a}+\vec{c} = \vec{b}+\vec{d} \Rightarrow \vec{c}-\vec{b} = \vec{d}-\vec{a}.$$

$$\Rightarrow \overrightarrow{BC} = \overrightarrow{AD}$$

Also, $\vec{c}-\vec{d} = \vec{b}-\vec{a} \Rightarrow \overrightarrow{DC} = \overrightarrow{AB}$

$\therefore$ the ~~quadrillo~~ quadrilateral ABCD is a parallelogram.

By vector method prove that the three medians of a triangle are concurrent.



Soln. Let, $\vec{a}, \vec{b}$ & $\vec{c}$ be the position vectors of the vertices A, B & C respectively of the triangle ABC with respect to an arbitrary origin. Then, the position vectors of the middle points D, E & F of the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$ & $\overrightarrow{AB}$ are $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{a}}{2}$ & $\frac{\vec{a}+\vec{b}}{2}$ respectively.

Now, the position vector of the point G which divides the median $\overline{AD}$ internally in the ratio 2:1 is

$$\frac{2 \times \frac{\vec{b}+\vec{c}}{2} + 1 \times \vec{a}}{2+1} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$$

Similarly, ~~we can show that the point which~~ the point which divide the median $\overline{BE}$ internally in the ratio 2:1 is

$$\frac{2 \times \frac{\vec{a}+\vec{c}}{2} + 1 . \vec{b}}{2+1} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$$

Similarly, the point which divide the median $\overline{CF}$ internally in the ratio 2:1 is

$$\frac{2 \times \frac{\vec{a}+\vec{b}}{2} + 1 . \vec{c}}{2+1} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$$

$\therefore$ three medians intersect at the common point G. Hence the three medians are concurrent & at the point of concurrence they are divided in the ratio 2:1