Asymptotic Notation

(1)

Growth of Function

	2^n	$n!$
$n=0 \rightarrow$	1	1
$n=1 \rightarrow$	2	1
$n=2 \rightarrow$	4	2
$n=3 \rightarrow$	8	6
$n=4 \rightarrow$	16	24
$n=5 \rightarrow$	32	120
$n=6 \rightarrow$	64	720

In long run $n!$ is always greater than 2^n.

So we can say that

$n! > 2^n \ \forall n \geq 4$

$n!$ has higher growth rate than 2^n.

(1) Prove that $5n+3 = O(n)$

$f(n) = 5n+3$ where $f(n) \leq c \cdot g(n)$

$5n+3 \leq 6 \cdot n$ Here $n \geq 3$

$\leq c \cdot n \rightarrow O(n)$.

(2) Prove that $6n^2+2n+3 = O(n^2)$

$f(n) = \cancel{6n^2+2n+3 \leq 11 \cdot n^2}$

$\cancel{\leq c \cdot n^2 = O(n^2)}$.

$f(n) = 6n^2+2n+3 \leq 7 \cdot n^2$ Here $n \geq 3$

$\leq c \cdot n^2 = O(n^2)$

* Big oh $\rightarrow$ Low Upper bound

Max^m

Big Omega $\rightarrow$ High Lower bound

Min^m

(3) Prove that $2n+2 = O(n)$

$f(n) = 2n+2$

$g(n) = n$

$2n+2 \leq 2n+n$

$\leq 3.n$

$\therefore f(n) = O(n) \ \forall n \geq 2$

So Here $C = 3$ and $n_0 = 2$

(4) $f(n) = 3n^2+5n+5$

$3n^2+5n+5 \leq 3n^2+5n+n$

$\leq 3n^2+\underline{6n}$

$\leq 3n^2+n*n$

$\leq 4n^2$

$\therefore f(n) = O(n^2)$

$\therefore 3n^2+5n+5 \leq 4.n^2$

$\downarrow$
c and $n \geq 6$

(1) $5n^2+3$

$f(n) = 5n^2+3 \leq 5n^2+n$

$\leq 6.n$

(5) $f(n) = n\log n + \log n + 10$

$n\log n + \log n + 10 \leq n\log n + \log n + \log_{10}^{10}$

$\leq n\log n + \log n + 10\log^{10}$

$\leq n\log n + \log n + 10\log n$

$\leq n\log n + 11\log n$

$n\log n + 11\log n \leq n\log n + n\log n.$

$\leq 2n\log n$

$\therefore f(n) = O(n\log n) \ \forall n \geq 11$

6. $f(n) = n^2 + n\log n$

$n^2 + n\log n \leq n^2 + n * n$

$\leq 2n^2$

$f(n) \leq c \cdot n^2 \rightarrow f(n) = O(n^2)$

7. $f(n) = 2n - 5$

$2n - 5 \leq 2n$

$\leq 2 \cdot n$

$f(n) = O(n)$ $\quad c = 2$, $n_0 = 5$

8. $f(n) = 2^n + n^2 + 2n + 5$

$2^n + n^2 + 2n + 5 \leq 2^n + n^2 + 2n + n$

$\leq 2^n + n^2 + 3n$ $\quad n \geq 5$

$2^n + n^2 + 3n \leq 2^n + n^2 + n * n$

$\leq 2^n + n^2 + n^2 \leq 2^n + 2n^2$ $\quad n \geq 3$

$2^n + 2n^2 \leq 2^n + 2 * 2^n$ $\quad n \geq 4$

$\leq 3 * 2^n$

$\therefore f(n) \leq 3 \cdot 2^n$ $\quad f(n) = O(2^n)$ $\quad c = 3$ $n_0 = 4$

Constant Function

1. $f(n) = 16$

$\rightarrow f(n) \leq 16 \cdot 1$ where $c = 16$, $n_0 = 0$

$\boxed{f(n) = O(1)}$

2. $f(n) = 27$

$f(n) \leq 27 \cdot 1$ where $c = 27$, $n_0 = 0$

$\boxed{f(n) = O(1)}$

3. $f(n) = 1627$

$f(n) \leq 1627 * 1$

where

$c = 1627$

$n_0 = 0$

$\boxed{f(n) = O(1)}$

Linear Function

① $f(n) = 3n + 5$

1st Procedure

$f(n) \leq c \cdot g(n)$

$3n + 5 \leq 3n + n \leq 4n$

$3n + 5 \leq 4 \cdot n \longrightarrow$ $c = 4$ $n \geq 5$

2nd procedure

$3n + 5 \leq 3n + 5n \leq 8n$

where $c = 8$ $n_0 = 1$ $\forall n \geq 1$

$\therefore \boxed{f(n) = O(n)}$

② $f(n) = 2n + 3$

$2n + 3 \leq 2n + n \leq 3n$ $n \geq 3$

$c = 3$, $n_0 = 3$

$\boxed{f(n) = O(n)}$

③ $f(n) = 7n + 5$

$7n + 5 \leq 7n + n \leq 8n$ $n \geq 8$

$c = 8$, $n_0 = 5$

or

$7n + 5 \leq 7n + 5n \leq 12n$

$7n + 5 \leq 12n$ $c = 12$ $n_0 = 1$

$\boxed{f(n) = O(n)}$

Quadratic Function

1. $f(n) = 27n^2+16n$

Taking $f(n) = 27n^2+16n$ $\quad \forall n^2 \geq 16$

$27n^2+16n \leq 27n^2+n^2$

$\leq 28 \cdot n^2$

$C = 28$

$n_0 = 16$

So $\boxed{f(n) = O(n^2)}$

2. $f(n) = 27n^2+16$

Taking $f(n) = 27n^2+16$ $\quad \forall n \geq 16$

$27n^2+16 \leq 27n^2+n$

$\rightarrow 27n^2+n \leq 27n^2+n^2 \leq 28n^2$

$\boxed{C=28 \\ n_0 = 1}$

So $\boxed{f(n) = O(n^2)}$

3. $f(n) = 10n^2+7$

$f(n) = 10n^2+7$ $\quad \forall n \geq 7$

$10n^2+7 \leq 10n^2+n$ $\quad \forall n \leq n^2$

$10n^2+n \leq 10n^2+n^2 \leq 11n^2$ $\quad [C = 11, n_0 = 1]$

$\therefore 10n^2+7 \leq 11 \cdot n^2$

$\boxed{f(n) = O(n^2)}$

(4) $f(n) = 27n^2+16n+25$

$f(n) = 27n^2+16n+25 \quad \forall n \geq 25$

$27n^2+16n+25 \leq 27n^2+16n+n$

$\leq 27n^2+16n$

$27n^2+16n \leq 27n^2+n^2 \leq 28 \cdot n^2 \quad (c=28 \quad n_0=17)$

$\therefore \boxed{f(n) = O(n^2)}$

Cubic Function

(1) $f(n) = 2n^3+n^2+2n$

$\forall n^2 \geq 2n$

$2n^3+n^2+2n \leq 2n^3+n^2+n^2$

$\leq 2n^3+2n^2$

$\forall n^3 \geq 2n^2$

$2n^3+2n^2 \leq 2n^3+n*n^2 \leq 2n^3+n^3 \leq 3n^3 \quad [c=3, n_0=2]$

$\therefore 2n^3+n^2+2n \leq 3 \cdot n^3$

$\therefore \boxed{f(n) = O(n^3)}$

(4)

(4) $f(n) = 4n^3 + 2n + 3$

$\therefore 4n^3 + 2n + 3 \leq 4n^3 + 2n + n \leq 4n^3 + 3n \quad (n \geq 3)$

$\boxed{n^3 \leq 3n}$

$\therefore 4n^3 + 3n \leq 4n^3 + n^3 \leq 5n^3 \quad [c = 5, n = 3]$

$\therefore \boxed{f(n) = O(n^3)}$

(5) $f(n) = 3n^3 + 4n$

$\boxed{n^3 \geq 4n}$

$3n^3 + 4n \leq 3n^3 + n^3 \leq 4 \cdot n^3$

$\therefore \boxed{f(n) = O(n^3)}$

(6) $f(n) = 5n^3 + n^2 + 3n + 2$

$\forall n \geq 2$

$5n^3 + n^2 + 3n + 2 \leq 5n^3 + n^2 + 3n + n \leq 5n^3 + n^2 + 4n$

$\forall n^2 \geq 4n$

$5n^3 + n^2 + 4n \leq 5n^3 + n^2 + n^2 \leq 5n^3 + 2n^2$

$\forall n^3 \geq 2n^2$

$5n^3 + 2n^2 \leq 5n^3 + n^3 \leq 6 \cdot n^3 \quad [c = 6, n = 2]$

$\boxed{f(n) = O(n^3)}$

Exponential Function

① $f(n) = 2^m + 6n^2 + 3n$

$\forall n^2 \geq 3n$

$2^n + 6n^2 + 3n \leq 2^n + 6n^2 + n^2 \leq 2^n + 7n^2$

$\forall 2^n \geq n^2 \quad (n \geq 4)$

$2^n + 7n^2 \leq 2^n + 7 * 2^n \leq 8 \times 2^m \quad [c=8, n_0 = 4]$.

$\boxed{f(n) = O(2^n)}$

② $f(n) = 4 * 2^n + 3n$

$\forall 2^m \geq n \quad (m \geq 1)$

$4 * 2^n + 3n \leq 4 * 2^n + 3 * 2^n = 7 * 2^n \quad (c = 7, n = 1)$

$\boxed{f(n) = O(2^n)}$

③ $f(n) = 5 * 2^n + 3n + 5$

$\forall n \geq 5$

$5 * 2^n + 3n + 5 \leq 5 * 2^n + 3n + n \leq 5 * 2^n + 4n$

$\forall 2^n \geq n \quad [n \geq 1]$

$5 * 2^n + 4n \leq 5 * 2^n + 4 * 2^n$

$\leq 9 * 2^m \quad [c = 9, n_0 = 1]$

$\therefore \boxed{f(n) = O(2^n)}$

4 $f(n) = 6*2^n + 6$

$\forall\ 2^n \geq 6$

$\therefore\ 6*2^n + 6 \leq 6*2^n + 2^n \leq 7*2^n$ $[c = 7, n_0 = 0]$

$\boxed{f(n) = O(2^n)}$

5 $f(n) = 3*2^n + 4n^2 + 5n + 3$

$\forall n \geq 3$

$3*2^n + 4n^2 + 5n + 3 \leq 3*2^n + 4n^2 + 5n + n$

$\leq 3*2^n + 4n^2 + 6n$

$\forall n \geq 6,\ n^2 \geq 6n$

$3*2^n + 4n^2 + 6n \leq 3*2^n + 4n^2 + n^2$

$\leq 3*2^n + 5n^2$

$n \geq 5 \quad n^2 \leq 2^n$

$3*2^n + 5n^2 \leq 3*2^n + 5*2^n \leq 8*2^n$ $[c = 8, n_0 = 5]$

$\boxed{f(n) = O(2^n)}$

Big - Omega -

① $f(n) = 2n^2 + 3n + 5$

Now $2n^2 + 3n + 5 \geq 2n^2 \quad \forall n \geq 1$

$\therefore \boxed{f(n) = \Omega(n^2)}$

② $f(n) = 2n - 3$

$2n - 3 \geq 2n - n \geq n \quad \forall n \geq 3$

$\boxed{f(n) = \Omega(n)}$

Constant Function

① $f(n) = 16$

$16 \geq 15 * 1 \rightarrow c = 15, n_0 = 1$

$\boxed{f(n) = \Omega(1)}$

② $f(n) = 27$

$27 \geq 26 * 1 \rightarrow c = 26, n_0 = 1$

$\boxed{f(n) = \Omega(1)}$

③ $f(n) = 1627$

$1627 \geq 1626 * 1 \rightarrow c = 1626, n_0 = 1$

$\boxed{f(n) = \Omega(1)}$

Linear Function

1. $f(n) = 3n+5$

$3n+5 > 3n \rightarrow \forall n \{c=3\}$

$\boxed{f(n) = \Omega(n)}$

2. $f(n) = 2n+3$

$2n < 2n+3$ for all n $\{c=2\}$

$\boxed{f(n) = \Omega(n)}$

3. $f(n) = 7n+5$

$7n < 7n+5 \quad \forall n \{c=7\}$

$\boxed{f(n) = \Omega(n)}$

Quadratic Function

1. $f(n) = 27n^2+16n$

$27n^2+16n > 27n^2 \quad \forall n \{c=27\}$

$\boxed{f(n) = \Omega(n^2)}$

2. $f(n) = 27n^2+16n+25$

$27n^2 < 27n^2+16n+25 \quad \forall n \{c=27\}$

$\boxed{f(n) = \Omega(n^2)}$

Cubic Function

① $f(n) = 2n^3 + n^2 + 2n$

$2n^3 < 2n^3 + n^2 + 2n \quad \forall n \quad \{c = 2\}$

$\boxed{f(n) = \Omega(n^3)}$

② $f(n) = 5n^3 + n^2 + 3n + 2$

$5n^3 < 5n^3 + n^2 + 3n + 2 \quad \forall n$

$c = 5$

$\boxed{f(n) = \Omega(n^3)}$

Exponential Function

① $f(n) = 2^n + 6n^2 + 3n$

$2^n < 2^n + 6n^2 + 3n \quad \forall n \quad \{c = 1\}$

$\boxed{f(n) = \Omega(2^n)}$

Big-Theta-

Constant Function

① $f(n) = 16$

$15 * 1 \leq f(n) \leq 16 * 1 \quad [c_1 = 15,\ c_2 = 16,\ n_0 = 0]$

$\boxed{f(n) = \Theta(1)}$

② $f(n) = 27$

$26 * 1 \leq f(n) \leq 27 * 1 \quad [c_1 = 26,\ c_2 = 27,\ n_0 = 0]$

$\boxed{f(n) = \Theta(1)}$

Linear Function

(1) $f(n) = 3n+5$

$3n \leq 3n+5 \quad \forall n \quad c_1 = 3$

also

$3n+5 \leq 4n \quad \forall n \geq 5 \quad [c_2 = 4, n_0 = 5]$

Thus -

$3n < 3n+5 \leq 4n \quad c_1 = 3, c_2 = 4, n_0 = 5$

so $\boxed{f(n) = \Theta(n)}$

(2) $f(n) = 2n+3$

$2n < 2n+3 \quad \forall n \quad c_1 = 2$

also

$2n+3 \leq 3n \quad \forall n \quad c_2 = 3$

thus -

$2n < 2n+3 \leq 3n \quad c_1 = 2, c_2 = 3, n_0 = 3$

$\boxed{f(n) = \Theta(n)}$

Quadratic Function

(1) $f(n) = 27n^2 + 16n$

$27n^2 < 27n^2 + 16n \quad \forall n, \ c_1 = 27$

also

$27n^2 + 16n \leq 28n^2 \quad \forall n \geq n_0, \ c_2 = 28, \quad n_0 = 16$

thus -

$27n^2 < 27n^2 + 16n \leq 28n^2 \quad c_1 = 27, c_2 = 28, n_0 = 16$

$\boxed{f(n) = \Theta(n^2)}$

(2) $f(n) = 27n^2+16n+25$.

$27n^2 < 27n^2+16n+25 \quad \forall n \geq n_0 \quad c_1=27$

also

$27n^2+16n+25 \leq 28n^2 \quad c_2=28, \quad n \geq n_0 = 17$

thus:

$27n^2 < 27n^2+16n+25 \leq 28n^2 \quad [c_1=27, c_2=28, n \geq n_0 = 17]$

$$\boxed{f(n) = \Theta(n^2)}$$

Cubic Function

(1) $f(n) = 2n^3+n^2+2n$

$2n^3 < 2n^3+n^2+2n \quad \forall n \geq n_0 \quad c_1=2$

also

$2n^3+n^2+2n \leq 3n^3 \quad \forall n \geq n_0 = 2, \; c_2=3$

thus.

$2n^3 < 2n^3+n^2+2n \leq 3n^3 \quad \forall n \geq n_0 = 2 \quad c_1=2, c_2=3$

$$\boxed{f(n) = \Theta(n^3)}$$

(2) $f(n) = 3n^3+4n$

$3n^3 < 3n^3+4n \quad \forall n \geq n_0 \quad c_1=3$

also.

$3n^3+4n \leq 4n^3 \quad \forall n \geq n_0 = 4, \; c_2=4$

thus -

$3n^3 < 3n^3+4n \leq 4n^3 \quad \forall n \geq n_0 = 4 \quad c_1=3, c_2=4$

$$\boxed{f(n) = \Theta(n^3)}$$

Exponential Function

1. $f(n) = 2^n + 6n^2 + 3n$

$2^n < 2^n + 6n^2 + 3n \quad \forall n \geq n_0 \quad c_1 = 1$

also

$2^n + 6n^2 + 3n \leq 8 * 2^n \quad \forall n \geq n_0 = 4, \; c_2 = 8$

thus

$2^n < 2^n + 6n^2 + 3n \leq 8 * 2^n \quad \forall n \geq n_0 = 4, \; c_1 = 1, \; c_2 = 8$

$$\boxed{f(n) = \Theta(2^n)}$$

2. $f(n) = 4 * 2^n + 3n$

$4 * 2^n \leq 4 * 2^n + 3n \quad \forall n \geq n_0 \quad c_1 = 4$

also

$4 * 2^n + 3n \leq 7 * 2^n \quad \forall n \geq n_0 = 1, \; c_2 = 7$

thus

$4 * 2^n < 4 * 2^n + 3n \leq \cancel{4 * 2^n + 3n}\; 7 * 2^n \quad \forall n \geq n_0 = 1$

$c_1 = 4, \; c_2 = 5$

$$\boxed{f(n) = \Theta(2^n)}$$

3. $f(n) = 3 * 2^n + 4n^2 + 5n + 3$

$3 * 2^n < 3 * 2^n + 4n^2 + 5n + 3 \quad \forall n \geq n_0 = 5, \; c_1 = 3$

also

$3 * 2^n + 4n^2 + 5n + 3 \leq 8 * 2^n \quad \forall n \geq n_0 = 5, \; c_2 = 8$

thus

$3 * 2^n < 3 * 2^n + 4n^2 + 5n + 3 \leq 8 * 2^n \quad \forall n \geq n_0 = 5, \; c_1 = 3, \; c_2 = 8$

$$\boxed{f(n) = \Theta(2^n)}$$