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Fourier Series

Definition

A trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ is said to be Fourier series for the function $f(x)$ defined on the closed interval $[-\pi,\pi]$ which is bounded and integrable in Riemann sense on the same closed interval $[-\pi,\pi]$ if and only if

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad ; k = 0,1,2,3,\ldots$$

$$\text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \quad , \quad k = 1,2,3,\ldots$$

Theorem

Let the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ be uniformly convergent to $f(x)$ on the closed interval $[-\pi,\pi]$, then the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ is a fourier series i.e

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad , \quad k = 0,1,2,3,\ldots$$

$$\text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \quad , \quad k = 1,2,3,\ldots$$

Proof:- Let $u_0(x) = \frac{1}{2}a_0$,

$$u_n(x) = a_n\cos nx + b_n\sin nx, \quad n = 1,2,3,\ldots$$

Then the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx),$$

reduces to the series $\sum_{n=0}^{\infty} u_n(x)$

Obviously each $u_n(x)$, $(n=0,1,2,\ldots)$ is continuous on the closed interval $[-\pi,\pi]$, Hence

$$\int_{-\pi}^{\pi} f(x)\,dx = \sum_{n=0}^{\infty}\int_{-\pi}^{\pi} u_n(x)\,dx$$

$$= \frac{1}{2}a_0\int_{-\pi}^{\pi}dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}(a_n\cos nx + b_n\sin nx)\,dx$$

$$= \frac{1}{2}a_0\cdot 2\pi + 0, \qquad \left[\because \int_{-\pi}^{\pi}\cos nx\,dx = \int_{-\pi}^{\pi}\sin nx\,dx = 0\right]$$

$$= \pi a_0.$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Again since the series $\sum_{n=0}^{\infty} u_n(x)$ is uniformly convergent on the closed interval $[-\pi,\pi]$, there exist a convergent series $\sum_{n=0}^{\infty} M_n$ of positive numbers, M_n being independent of x such that

$$|u_n(x)| \le M_n, \quad n=0,1,2,\dots, \text{ and } -\pi \le x \le \pi.$$

Now $f(x) = \sum_{n=0}^{\infty} u_n(x), \quad -\pi \le x \le \pi$

so, $f(x)\cos kx = \sum_{n=0}^{\infty} u_n(x)\cos kx, \qquad -\pi \le x \le \pi,\ k=1,2,3,\dots$

Since for $n=1,2,3,\dots$, and $-\pi \le x \le \pi$

$$|u_n(x)\cos kx| \le |u_n(x)| \le M_n$$

and since the series $\sum_{n=0}^{\infty} M_n$ of positive number is convergent, so $\sum_{n=0}^{\infty} u_n(x)\cos kx$ is uniformly convergent on the closed interval $[-\pi,\pi]$ for $k=1,2,3,\dots$.

So, for $k=1,2,3,\dots$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \sum_{n=0}^{\infty}\int_{-\pi}^{\pi} u_n(x)\cos kx\,dx$$

$$= \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos kx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right]$$

$$= a_k\pi, \qquad \left[\because \int_{-\pi}^{\pi}\cos nx\cos kx\,dx = 0,\ n\ne k \atop = \pi,\ n=k,\right.$$

$$\text{and} \int_{-\pi}^{\pi}\sin nx\cos kx\,dx = 0, \text{ for } n,k=1,2,3,\dots$$

$$\therefore a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

Proceeding similarly it can be shown that,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx.$$

Thus we have,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n=0,1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n=0,1,2,\dots$$

Consequently the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

is a Fourier Series for the function $f(x)$.

Theorem: Let $f(x)$ be bounded and integrable on the closed interval $[-a,a]$ $(a>0)$. Also let $f(x)$ be monotonic on the closed interval $[0,\alpha]$, where $0<\alpha<a$.

Then,

$$\lim_{n\to\infty}\int_0^a f(x)\frac{\sin nx}{x}\,dx = f(0+)\int_0^\infty \frac{\sin nx}{x}\,dx.$$

where $f(0+)$ denotes the limit of $f(x)$ as x tends to zero from the right of zero.

Drichlet's Conditions:

A function $f(x)$ will be said to satisfy Drichlet's conditions in an interval $-\pi\le x\le \pi$ in which it is defined when it is subject to one of the two following conditions:-

(i) $f(x)$ is bounded in $-\pi\le x\le\pi$, and the interval can be broken up into a finite number of open partial intervals in each of which $f(x)$ monotonic.

(ii) $f(x)$ has a finite number of points of infinte discontinuity in the interval. When arbitrary small neighbourhoods of these points are excluded, $f(x)$ is bounded in the remaining interval and this can be broken up into a finite number of open intervals in each of which $f(x)$ is monotonic.

Convergence When $f(x)$ satisfies Drichlet's condition in $-\pi\le x\le\pi$, the Fourier series corresponding to $f(x)$ converges to $f(x)$ at any point x in $-\pi\le x\le\pi$ where $f(x)$ is continuous and converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when there is an ~~arbitrary~~ ordinary discontinuity at the point. In particular at $x=\pi$ and $x=-\pi$, it converges to

$$\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\} \text{ when } f(-\pi+0) \text{ and } f(\pi-0) \text{ exist.}$$

Theorem Let f be bounded and integrable in $[-\pi,\pi]$, and let it be possible to divide $[-\pi,\pi]$ into a finite number of open subintervals, in each of which f is monotonic. Then the Fourier series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\xi + b_n\sin n\xi) \text{ for the function } f(x) \text{ converges}$$

to $\frac{1}{2}[f(\xi-0)+f(\xi+0)]$, $-\pi<\xi<\pi$ and converges to

$\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$ when $\xi=-\pi$ or $+\pi$.

Proof

First we prove the following Lemma.

Lemma If f is bounded and integrable in every interval and is periodic with 2π as its period, then

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(a+x)\,dx,$$

a being any number whatever.

Proof of the Lemma

Putting $a+x=y$, we have

$$\int_{-\pi}^{\pi} f(a+x)\,dx = \int_{a-\pi}^{a+\pi} f(y)\,dy$$

$$= \int_{a-\pi}^{-\pi} f(y)\,dy + \int_{-\pi}^{\pi} f(y)\,dy + \int_{\pi}^{a+\pi} f(y)\,dy \quad \text{——— (1)}$$

Now $\int_{a-\pi}^{-\pi} f(y)\,dy = \int_{a+\pi}^{\pi} f(z-2\pi)\,dz$

Putting $y = z-2\pi$, $dy = dz$

y	$-\pi$	$a-\pi$
z	π	$a+\pi$

$$= -\int_{\pi}^{a+\pi} f(z)\,dz$$

$$= -\int_{\pi}^{a+\pi} f(y)\,dy$$

Hence, from (1),

$$\int_{-\pi}^{\pi} f(a+x)\,dx = \int_{-\pi}^{\pi} f(y)\,dy$$

$$= \int_{-\pi}^{\pi} f(x)\,dx.$$

Proof of the main theorem

Since $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

So, we have,

$$\frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n\cos n\xi + b_n\sin n\xi)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,[\cos nx\cos n\xi + \sin nx\sin n\xi]\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[1 + 2\sum_{n=1}^{m}\cos n(x-\xi)\right]dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\xi)\left[1+2\sum_{n=1}^{m}\cos nx\right]dx, \quad \text{by lemma.}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} f(-2y+\xi)\cdot\frac{\sin(2m+1)y}{\sin y}dy + \frac{1}{\pi}\int_0^{\pi/2} f(2y+\xi)\frac{\sin(2m+1)y}{\sin y}dy.$$

by putting $x=-2y$ in the first integral and $x=2y$ in the second integral.

$$= \frac{1}{\pi}\left\{\frac{\pi}{2}f(\xi-0)+\frac{\pi}{2}f(\xi+0)\right\}$$

$$= \frac{f(\xi+0)+f(\xi-0)}{2}, \quad \text{as } m\to\infty.$$

Half range series

A. Cosine Series :-

Let $f(x)$ satisfies Drichlets conditions in $0\le x\le\pi$, then

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$ and $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$

is called fourier Cosine Series corresponding to $f(x)$ in the interval $0\le x\le\pi$. The series is equal to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at every x in $0<x<\pi$ where $f(x+0)$ and $f(x-0)$ exists, and is equal to ~~the~~ $f(0+0)$ ~~and~~ at $x=0$ and equal to $f(\pi-0)$ at $x=\pi$, provided both the limits exist.

If moreover $f(x)$ be continuous in the interval $[0,\pi]$, the cosine series represents $f(x)$ in the closed interval $0\le x\le\pi$.

Proof

We define an even function $F(x)$ in $[-\pi,\pi]$ which is ~~ind~~ identical with $f(x)$ in $[0,\pi]$. Thus

$F(x)=f(x)$ in $[0,\pi]$ and $F(x)=F(-x)=f(-x)$ in $[-\pi,0]$.

clearly $F(x)$ will satisfy the Drichlets Conditions in $[-\pi,0]$, if does so in $[0,\pi]$. Thus we see that the sum of the series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\sin nx \text{ where } a_n=\frac{\pi}{2}\int_0^{\pi}F(x)\cos nx\,dx,$$

$$=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx, \quad n=0,1,2\ldots$$

is equal to $\frac{1}{2}[F(x+0)+F(x-0)]=\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and π.

At $x=0$, the sum of the series $=\frac{1}{2}[F(0+0)+F(0-0)]$

$$=F(0+0)=f(0+0).$$

Similarly, we see that,

the sum of the series is $f(\pi-0)$ at $x=\pi$.

B. Sine Series :-

Let $f(x)$ satisfies Drichlets Conditions in $0\leq x\leq \pi$, then $\sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$ represents $f(x)$ in Fourier sine Series in $[0,\pi]$. The series is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x in $0<x<\pi$ when $f(x+0)$ and $f(x-0)$ exist and is equal to 0 when $x=0$ and $x=\pi$.

Proof

Here we define an odd function $f(x)$ in $[-\pi,\pi]$, which is identical with $f(x)$ in $[0,\pi]$.

Proceeding as above we can prove the sine series.

Other forms of Fourier Series

A. In the interval $-l\leq x\leq l$

The Fourier series corresponding to $f(x)$ in any interval $-l\leq x\leq l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx$.

B. In the interval $0\leq x\leq 2\pi$

The Fourier Series corresponding to $f(x)$ in $[0,2\pi]$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\, dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$.

C. In the interval $a\leq x\leq b$

The Fourier series corresponding to $f(x)$ in $[a,b]$ is

$$\frac{1}{2}a_0 + \sum\left(a_n\cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2\pi n x}{b-a}\right)$$

where $a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\, dx$

$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\, dx$.

1. Show that,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right)$$

Let $f(x) = |x|$.

So, $f(x)$ is bounded and integrable in $-\pi \le x \le \pi$, since it in continous here. Also $f(x)$ is monotonic in each of the closed interval $[-\pi, 0]$ and $[0, \pi]$. So, $f(x)$ satisfies Dirichlet's conditions in $[-\pi, \pi]$. Thus the Fourier series for the function $f(x)$ converges to $f(x)$ on the closed interval $[-\pi, \pi]$.

Let, $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier Series of the function $f(x)$. Then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{——} (1)$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx$

$$= \frac{-1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{-1}{2\pi}\left[x^2\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[x^2\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 + \pi^2\right]$$

$$= \frac{2\pi^2}{2\pi}$$

$$= \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cdot \cos nx\,dx$$

$= -\frac{4}{\pi n^2}$, when n is odd number (H.W)

0, when n is even number.

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x| \sin nx\,dx$

$= 0$, [since $|x| \sin nx$ is an odd function.]

Then (1) reduces to,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right).$$

2. Find a Fourier Series representing $f(x)$ in $-\pi \le x \le \pi$, when

$$f(x) = 0, \quad -\pi \le x \le 0$$
$$= \frac{\pi}{4}x, \quad 0 < x \le \pi.$$

and deduce that,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Given that,

$$f(x) = 0 \ , \ -\pi \le x \le 0,$$
$$= \frac{\pi x}{4}, \ 0 < x \le \pi.$$

The function $f(x)$ is bounded and integrable in $[-\pi,\pi]$ as it is continuous there. Also $f(x)$ is monotonic in each of the closed interval $[-\pi,0]$, $[0,\pi]$. So, $f(x)$ satisfies Dirichlet's conditions in $[-\pi,\pi]$. Thus the ~~function~~ fourier series for the function $f(x)$ converges to $f(x)$ on the closed interval $[-\pi,\pi]$.

Let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the fourier series of the function $f(x)$.

Then,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{---} (1).$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{4}x\,dx$$

$$= \frac{\pi^2}{8}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx\,dx$$

$$= \frac{\pi}{4\pi}\int_{0}^{\pi} x \cos nx\,dx$$

$$= -\frac{1}{2n^2}, \quad n \text{ is odd}.$$

$$= 0, \quad n \text{ is even}.$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx\,dx$

$$= \frac{\pi}{4\pi}\int_{0}^{\pi} x \sin nx\,dx$$

$$= -\frac{\pi}{4n}, \quad n \text{ is even}$$

$$= \frac{\pi}{4n}, \quad n \text{ is odd}.$$

$\therefore$ (1) reduces to,

$$f(x) = \frac{\pi^2}{16} - \frac{1}{2}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right) + \frac{\pi}{4}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right). \quad \text{---} (2).$$

This is the required fourier Series.

Putting $x=0$, in (2), we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

3. Expand in a Series of sines and cosines of multiples of x, the function given by,

$$f(x) = x-\pi, \quad \text{when } -\pi < x < 0,$$
$$= \pi - x, \quad \text{when } 0 < x < \pi.$$

What is the sum of the series for $x = \pm\pi$ and $x = 0$?

Here $f(x)$ is not defined at $x = 0, \pm\pi$, where it can be defined in any manner. for convenience, let us take $f(x) = x-\pi$ at $x = -\pi, 0$ and $f(x) = \pi - x$ at $x = \pi$. Thus the function $f(x)$ being continuous in $-\pi \le x \le \pi$ except at $x=0$, where there is an arbitrary discontinuity, is bounded and integrable there. Further $f(x)$ is monotonic in each of the open interval $-\pi < x < 0$ and $0 < x < \pi$. Then $f(x)$ satisfies Dirichlet's conditions in $[-\pi, \pi]$.

Now, $a_0 = -\pi$,

$$a_n = \frac{2[1-(-1)^n]}{n^2 \pi}$$

and $b_n = \dfrac{2[1-(-1)^n]}{n}$ (H.W)

Thus the Fourier series corresponding to $f(x)$ is

$$-\frac{1}{2}\pi + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right],$$

$-\pi < x < 0, \; 0 < x < \pi.$ ——— (1).

Also $f(x)$ is continuous in $-\pi < x < 0$, $0 < x < \pi$.

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 4\left[\sin x + \frac{\sin 3x}{3} + \cdots\right]$$

When $x=0$ is a point of discontinuity of $f(x)$, therefore, the sum of the series for $x=0$ is

$$\frac{1}{2}[f(+0) + f(-0)] = \frac{1}{2}(\pi - \pi) = 0.$$

Hence we conclude that,

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 4\left[\sin x + \frac{\sin 3x}{3} + \cdots\right]$$

$-\pi < x < \pi$

Putting $x = 0$ in (1), we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad \text{——— (2).}$$

for $x = \pm\pi$, the sum of the series $= \frac{1}{2}[f(\pi-0) + f(-\pi+0)]$ ~~eq or~~

$$= \frac{1}{2}[0 + (-2\pi)] = -\pi$$

Putting $x = \pm\pi$, we obtain the ~~ess~~ result,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

4) If $f(x) = -\frac{\pi}{4}$, when $-\pi < x < 0$,

$= \frac{\pi}{4}$, when $0 < x < \pi$

$f(-\pi) = f(0) = f(\pi) = 0$ and $f(x+2\pi) = f(x)$ for all x, then show that for all x,

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots$$

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Here $f(x)$ being continuous in $-\pi \leq x \leq \pi$ except at $x = 0, +\pi, -\pi$, where there are an arbitrary discontinuity, is bounded and integrable there. Further $f(x)$ is monotonic in each of the open intervals ~~$-\pi \leq x <$~~ $-\pi < x < 0$ and $0 < x < \pi$. Thus $f(x)$ satisfies ~~Dir~~ Dirichlet's conditions in $-\pi \leq x \leq \pi$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= 0$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= 0$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} \frac{(-\pi)}{4}\cdot \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi}{4}\cdot \sin nx\,dx$

$= \frac{1}{4n}\left[\cos nx\right]_{-\pi}^{0} + \frac{1}{4n}\left[\cos n\pi\right]_{0}^{\pi}$

$= -\frac{1}{2n}\cos n\pi + \frac{1}{2n}$

$= +\quad 0, \quad n$ is even

$= \frac{1}{n}, \quad n$ is odd.

Thus the fourier series corresponding to $f(x)$ is

$$\sum_{n=1}^{\infty} \frac{1}{n}\cdot\{1-(-1)^n\}\sin nx$$

$$= \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots, \quad -\pi < x < 0,\ 0 < x < \pi. \qquad \text{——— (2)}.$$

also $f(x)$ is continuous in $-\pi < x < 0$, $0 < x < \pi$

$\therefore f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots$

When $x = 0$, is a point of discontinuity of $f(x)$, therefore, the sum of the for $x = 0$ is

$$\frac{1}{2}[f(0+) + f(0-)] = \frac{1}{2}\left[\pi/4 - \pi/4\right] = 0.$$

Similarly, for $x = \pm\pi$, the sum of the series

$$\frac{1}{2}[f(\pi-0) + f(-\pi+0)] = 0 = f(\pm\pi).$$

Also,

$f(x+2\pi) = f(x)$, for all x.

Hence we conclude that,

~~$f(x) = \sin x$~~

$$f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \quad \text{——(2)}.$$

Putting $x = \pi/2$, in (2), we get,

$$f(\pi/2) = \sin \pi/2 + \frac{1}{3}\sin 3\pi/2 + \frac{1}{5}\sin 5\pi/2 + \cdots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

5. Expand a fourier series $x+x^2$ in $-\pi<x<\pi$ and deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Let $f(x) = x + x^2$ in $-\pi<x<\pi$.

We may define $f(x)$ at the end points $x = \pm\pi$ ~~arbitrarily~~ arbitrarily. But for convenience we take $f(x) = x+x^2$ in $-\pi \le x \le \pi$. Now $f(x)$ is bounded and integrable in $[-\pi,\pi]$ because of its continuity in the same closed interval. further, $f'(x) = 1+2x$, so that $f'(x)>0$ for $x>-1/2$ and $f'(x)<0$ for $x<-1/2$. Thus ~~f~~ $f(x)$ is monotonic decreasing in $-\pi<x<-1/2$ and monotonic increasing in $-1/2<x\le\pi$. Hence $f(x)$ satisfies Dirichlet's conditions in $[-\pi,\pi]$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{2}{3}\cdot\pi^2.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}, \text{ when } n \text{ is even}$$

$$= -\frac{4}{n^2}, \text{ when } n \text{ is odd}.$$

Similarly, $b_n = -2/n$, when n is even

$= 2/n$, when n is odd.

Thus Fourier series corresponding to $f(x)$ is

$$\frac{\pi^3}{3} + 4\left\{-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right\} + 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Also $f(x)$ is continuous in $-\pi<x<\pi$,

$$\therefore x+x^2 = \frac{\pi^3}{3} + 4\left[-\cos x + \frac{\cos 2x}{2} - \frac{\cos 3x}{3} + \cdots\right] + 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right].$$

——(i)

Next at $x = \pm\pi$, the sum of the series,

$$= \frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right]$$

$$= \frac{1}{2}\left[\cancel{-\pi} + \pi^2 + \cancel{\pi} + \pi^2\right]$$

$$= \pi^2.$$

Putting $x = \pm\pi$, in (i), we get,

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right)$$

or, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

6) Find the fourier series which represents $|\sin x|$ in $-\pi \le x \le \pi$.

H.W

7) Find a series of Cosines of multiples of x which will represents $f(x) = x$ in the closed interval $0 \le x \le \pi$.

Observe that $f(x)$ is bounded and integrable in $0 \le x \le \pi$, since it is continuous there. Further $f'(x) = 1 > 0$ indicates that $f(x)$ is monotonic increasing in the entire interval. Thus $f(x)$ satisfies Dirichlets conditions in $0 \le x \le \pi$. Hence it can be expanded in Fourier Cosine series in the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$.

and $a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2}{n^2\pi}(\cos n\pi - \pi) = \begin{cases} 0 & , n \text{ even} \\ \frac{-4}{n^2\pi} & , n \text{ odd}. \end{cases}$

So the Fourier series corresponding to $f(x)$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\left\{ \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right\}.$$

Also $f(x) = x$ is continuous in $0 \le x \le \pi$.

$\therefore$ We get, $x = \frac{\pi}{2} - \frac{4}{\pi}\left\{ \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right\}$, in $0 \le x \le \pi$.

Deduction, At $x = 0$

$$0 = \pi/2 - \frac{4}{\pi}\left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right\}$$

$$\therefore 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

8) Find a series of Sines of multiples of x which will represent x in $0 \le x < \pi$.

Let $f(x) = x$ in $0 \le x < \pi$.

We may define $f(x)$ at the end point π arbitrarily. But for convenience let us define $f(x) = x$ in $0 \le x < \pi$. Then $f(x)$ is bounded and integrable in the interval $0 \le x < \pi$, since it is continuous there. Since $f'(x) = 1 > 0$, then $f(x)$ is monotone increasing in the interval $0 \le x < \pi$. Thus $f(x)$ satisfies Dirichlets conditions in $0 \le x < \pi$. ~~then~~ Hence Fourier Sine series corresponding to $f(x)$ is

$$\sum_{n=1}^{\infty} b_n \sin nx.$$

$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \frac{-2}{\pi n}\left[x \cos nx\right]_0^{\pi} + \frac{2}{\pi n^2}\left[\sin nx\right]_0^{\pi}$

$= -\frac{2}{n}$, n is even

$= \frac{2}{n}$, n is odd.

The Fourier Sine series corresponding to $f(x)$ is

$$\frac{2\pi}{\pi}\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

Also $f(x)$ is continuous in $0 \le x < \pi$, then the series is,

$$x = 2\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

Deduction, At $x=0$, the sum of the series is $0 = f(0)$ for $0 \le x < \pi$, and at $x = \pi$, the sum of the series is $0 \ne f(\pi)$.

$$\therefore x = 2\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}.$$

9) Expand $f(x)$ in Fourier Cosine Series in $0 \le x \le \pi$ where $f(x) = \sin x,\ 0<x<\pi$.

Ans. $f(x) = \frac{2}{\pi}\left\{1 - 2\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right\}\right.$

10) Expand $f(x)$ in Fourier Sine series in $0 \le x \le \pi$, where

$$f(x) = \tfrac{1}{4}\pi x, \quad 0 \le x \le \pi/2$$
$$= \tfrac{1}{4}\pi(\pi - x), \quad \pi/2 \le x \le \pi.$$

Ans. $f(x) = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \cdots$

11) Develope $f(x)$ in Fourier series in $-\pi < x < \pi$ if

$$f(x) = 0 \quad \text{for } -\pi < x < 0$$
$$= \pi \quad \text{for } 0 < x < \pi.$$

Ans. $\frac{1}{2}\pi + 2\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}$.

12) Obtain the Fourier series corresponding to the function $f(x)$ where

$$f(x) = 0 \quad \text{where } -\pi < x < 0$$
$$= 1 \quad \text{where } 0 < x < \pi.$$

Ans. $\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$.

13) Expand $f(x) = x,\ 0 < x < 2$, in a Fourier Cosine series.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$, here $l = 2$.

where $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right], \text{ if } n \ne 0.$$

$\therefore a_n = -8/n^2\pi^2$ if $n = 1, 3, 5, \ldots$ and $a_n = 0$ if $n = 2, 4, 6, \ldots$

Also, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2$.

(d) $\therefore f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \cdots\right]$.

The L^2-theory of fourier Series

A function f is said to be square integrable over $[0, 2\pi]$ when f is said measurable and $\int_0^{2\pi} f^2 dx < \infty$. In this case, we also write $f^2 \in L[0, 2\pi]$ or $f \in L^2[0, 2\pi]$.

Parseval's Identity for fourier Series

Let the fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ of $f(x)$ converge uniformly to $f(x)$ at every point of the interval $(0, 2\pi)$ and let $f(x) \in L[0, 2\pi]$. Then

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ——— (1).

Multiplying (1) by $f(x)$ and then integrating, we get,

$$\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_0^{2\pi} f(x)dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi} f(x)\cos nx\, dx + b_n\int_0^{2\pi} f(x)\sin nx\, dx\right]$$

Making use of the fact that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

we get,

$$\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0}{2}\pi a_0 + \sum_{n=1}^{\infty}(\pi a_n^2 + \pi b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Bessel's Inequality

We now first proof

$$\frac{a_0^2}{2} + \sum_{n=1}^{m}(a_n^2 + b_n^2) \leq \frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx.$$

where a_n and b_n are the ~~fourier~~ fourier coefficients of $f(x)$, $f(x)$ being piecewise continuous in $(-l, l)$.

Proof We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{——— (1)}$$

where $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$ $\Big\}$ ——— (2)

Take $S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$ ——— (3).

Evidently,

$$[f - S_m^2] \geq 0$$

Expand $f(x)$ in Fourier cosine series in $0 \le x \le \pi$ where $f(x) = \sin x$, $0 < x < \pi$.

Solution : Here $f(x)$ is not defined at $x=0, \pi$ where it can be defined in any manner. For convenience let us take $f(x) = \sin x$ at $x=0, \pi$. Thus $f(x)$ being continuous in $0 \le x \le \pi$, is bounded and integrable there. Further $f(x)$ is monotonic in each of the interval $0 < x < \pi/2$ and $\pi/2 \le x < \pi$. Thus $f(x)$ satisfies Dirichlet's condition in $[0, \pi]$ fourier cosine series corresponding to $f(x)$ is $f = \sin x$ is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}(-\cos x)_0^{\pi} = \frac{4}{\pi}$

and, $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\times\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\times\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\times\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2}{n^2-1}\cos n\pi - \frac{2}{n^2-1}\right)$$

$$= -\frac{1}{\pi}\times\frac{2}{n^2-1}\cdot(\cos n\pi + 1)$$

$$= -\frac{2}{\pi(n^2-1)}\cdot\{(-1)^n+1\} = -\frac{4}{\pi(n^2-1)}, \text{ when } n \text{ is even}$$

$$= 0, \text{ when } n \text{ is odd}.$$

Thus, the fourier series corresponding to $f(x)$ is,

$$\frac{2}{\pi} - \frac{4}{\pi}\cdot\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right)$$

Also, $f(x)$ is continuous in $0 < x < \pi$.

$\therefore f(x) = \frac{2}{\pi}\left[1 - 2\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right)\right]$

Next at $x=0$, the sum of series is,

$\frac{1}{2}[f(+0) + f(-0)] = \frac{1}{2}\times 0 = 0$

and at π the sum of series is 0. Hence, $x = \frac{2}{\pi}\left\{1 - 2\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right)\right\}$

7. Let $f(x) = \frac{1}{4}\pi x$, when $0 \le x \le \pi/2$

$= \frac{1}{4}\pi(\pi - x)$, when $\pi/2 \le x \le \pi$

Find: (1) a series of cosines of multiples of x which will represent $f(x)$ in the interval $0 \le x \le \pi$

(2) a series of sines of multiples of x which will represent $f(x)$ in the interval

Solution: Here $f(x)$ is bounded, integrable and piecewise monotonic in $[0, \pi]$. Thus, $f(x)$ satisfies Dirichlet's condition in $[0, \pi]$.

(1) Hence fourier cosines series corresponding to $f(x)$ is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{4}\pi x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{1}{4}\pi(\pi - x)\,dx$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{1}{2}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\cdot\frac{\pi^2}{8} + \frac{1}{2}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{8}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{4}\pi x \cdot \cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{1}{4}\pi(\pi - x)\,dx$$

$$= \frac{1}{2}\left\{\left[\frac{x \sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\,dx\right\} + \frac{1}{2}\left[(\pi - x)\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} + \frac{1}{2}\int_{\pi/2}^{\pi} \frac{\sin nx}{n}\,dx$$

$$= \frac{1}{2n}\left\{\left(\frac{\pi}{2}\cdot\sin\frac{n\pi}{2}\right) + \left[\frac{\cos nx}{n}\right]_0^{\pi/2}\right\} + \frac{1}{2n}\left\{\left(0 - \frac{\pi}{2}\sin\frac{n\pi}{2}\right) + \left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{2n}\times\left\{\frac{\pi}{2}\sin\frac{n\pi}{2} + \frac{1}{n}\cos\frac{n\pi}{2} - \frac{1}{n}\right\} + \frac{1}{2n}\times\left\{-\frac{\pi}{2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi + \frac{1}{n}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{2n^2}\cos n\pi - \frac{1}{2n^2} = \frac{1}{n^2}\times\left\{\cos\frac{n\pi}{2} - \frac{1}{2}\left[1 + (-1)^n\right]\right\}$$

So, the fourier cosine series corresponding to $f(x)$ is,

$$\frac{\pi^2}{16} - 2\left(\frac{1}{2^2}\cos 2x + \frac{1}{4^2}\cos 4x + \dots\right)$$

(2) Hence, the fourier sine series corresponding to $f(x)$ is,

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{4}\pi x \sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{1}{4}\pi(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2} \cos nx\,dx\right\} + \frac{1}{2}\left\{\left[-(\pi - x)\frac{\cos nx}{n}\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}(-1)\left(-\frac{\cos nx}{n}\right)dx\right.$$

or, $b_n = \frac{1}{2n}\times\left\{\left(-\frac{\pi}{2}\cos\frac{n\pi}{n}+0\right)+\left[\frac{\sin nx}{n}\right]_0^{\pi/2}\right\}+\frac{1}{2n}\times\left\{\left(0+\frac{\pi}{2}\cos\frac{n\pi}{2}\right)-\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right\}$

$= \frac{1}{2n}\left(-\frac{\pi}{2}\cos n\pi/2+1/n\sin\frac{n\pi}{2}\right)+\frac{1}{2n}\left(\frac{\pi}{2}\cos n\pi/2+1/n\sin n\pi/2\right)$

$= \frac{1}{n^2}\cdot\sin\frac{n\pi}{2}$

So, the fourier sine series corresponding to $f(x)$ is,

$\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots$

Also, $f(x)$ is continuous in $0\leq x\leq \pi$, so

$f(x) = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots$

3. Develope $f(x)$ is fourier series in $-\pi<x<\pi$ if

$f(x) = 0$, for $-\pi<x<0$

$= \pi$, for $0<x<\pi$

Solution: Here $f(x)$ is not defined at $x=0$.

$a_0 = 1/\pi\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}\pi dx = \pi$

$a_n = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\, dx = 1\times 0 = 0$

$b_n = \frac{1}{\pi}\cdot\pi\int_0^{\pi}\sin nx dx = \left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{\cos n\pi}{n}+\frac{1}{n}$.

$= \frac{(-1)^{n+1}}{n}+\frac{1}{n} = 0$, if n is even

$= 2/n$, if n is odd.

Therefore,

$f(x) = \pi/2 + 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$, in $-\pi<x<0$ and $0<x<\pi$.

When $x=0$, which is a point of discontinuity of $f(x)$ we have,

$\frac{1}{2}[f(+0)+f(-0)] = \frac{1}{2}(\pi+0) = \pi/2 = f(0)$

Hence, $f(x) = \pi/2 + 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$, in $-\pi<x<\pi$.

9. Obtain the fourier series corresponding to the function $f(x)$ where,

$$f(x)=0 \ , \text{ when } -\pi<x<0$$
$$=1 \ , \text{ when } 0\le x\le \pi$$

Solution: Here $f(x)$ is not defined at $x=-\pi$ where it can be defined in any manner. For convenience let us take $f(x)=0$ at $x=-\pi$. Thus $f(x)$ being continuous in $-\pi\le x\le \pi$ except at $x=0$, which is an ordinary point of discontinuity, and it is bounded and integrable there. Further $f(x)$ is piecewise monotonic. Thus $f(x)$ satisfies Dirichlet's conditions in $[-\pi,\pi]$.

Now,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 1.$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\times(-\cos n\pi+1)$$

$$= \frac{1}{n\pi}\times\{(-1)^{n+1}+1\} = \frac{2}{n\pi}, \text{ if } n \text{ is odd}$$
$$= 0, \text{ if } n \text{ is even}$$

So the fourier series corresponding to $f(x)$ is,

$$\tfrac{1}{2} + \tfrac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right), \text{ in } [-\pi,\pi].$$

10. Obtain a fourier series of a periodic function f with period 2π defined as

$$f(x) = -1, \ -\pi<x<0$$
$$=0, \quad x=0$$
$$=1, \ 0<x<\pi$$

Deduce from the series and obtain $\pi/4 = 1 - 1/3 + 1/7 - \cdots$. What is the value of the series at $x=0, \pm\pi$?

Solution:

Here $f(x)$ is not defined at $x=\pm\pi$, where it can be defined in any manner. For convenience let us take $f(x)=-1$ at $x=-\pi$ and $f(x)=1$ at $x=\pi$. Thus $f(x)$ being continuous in $-\pi\le x\le\pi$ except at $x=0$, is bounded and integrable there. Further $f(x)$ is monotonic in each open interval $(-\pi,0)$ and $(0,\pi)$. Thus $f(x)$ satisfies Dirichlet's conditions in $[-\pi,\pi]$.

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{-\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi n}\times\left\{-(-1+\cos n\pi) + (-\cos n\pi + 1)\right\}$$

$$= \frac{2}{n\pi}(1-\cos n\pi) = \frac{4}{n\pi} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

Thus the fourier series corresponding to $f(x)$ is,

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right), \text{ in } -\pi < x < 0 \text{ and } 0 < x < \pi.$$

Also, $f(x)$ is continuous in $[-\pi, \pi]$. Therefore,

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

When $x=0$ is a point of discontinuity of $f(x)$, the sum of the series for $x=0$ is,

$$\tfrac{1}{2}\{f(0+0) + f(0-0)\} = \tfrac{1}{2}\times(1-1) = 0 = f(0)$$

Similarly, for $x = \pm\pi$ the sum of series is,

$$\tfrac{1}{2}\{f(\pi-0) + f(-\pi+0)\} = \tfrac{1}{2}(1-1) = 0$$

Also since f is a periodic function with period 2π, so

$$f(x \pm 2\pi) = f(x)$$

Hence we conclude that

$$f(x) = \frac{4}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\} \quad \forall x \quad \text{——} (1)$$

Putting $x = \pi/2$ in (1) we have,

$$f(\pi/2) = 4/\pi\left(1 - 1/3 + 1/5 - 1/7 + \cdots\right)$$

$$\text{or, } 1 = 4/\pi\left(1 - 1/3 + 1/5 - 1/7 + \cdots\right)$$

$$\text{or, } \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$$

11. Find the fourier series for $f(x)$ on $0 < x < 2\pi$, where $f(x) = (x-\pi)^2$ and $f(x) = \pi^2$, $\pi \le x \le 2\pi$. Hence deduce the value of $\Sigma 1/n^2$.

12. Find the fourier series of sine of multiples of x, which will represent the function in the interval $0 < x < \pi$, where.

$$f(x) = \pi/3 \text{, when } 0 < x < \pi/3$$
$$= 0 \text{, when } \pi/3 < x < 2/3\pi$$
$$= -\pi/3 \text{, when } 2/3\pi < x < \pi.$$

Find the sum of the series at the points $2\pi/3$ and π.

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/3} \pi/3 \sin nx\,dx + \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} 0.\sin nx\,dx + \frac{2}{\pi}\int_{2\pi/3}^{\pi} (-\pi/3).\sin nx\,dx$$

$$= 2/3\left[-\frac{\cos nx}{n}\right]_0^{\pi/3} - 2/3\left[-\frac{\cos nx}{n}\right]_{2\pi/3}^{\pi}$$

$$= \frac{2}{3n}\left(\cos n\pi - \cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3} + 1\right)$$

Therefore the corresponding fourier series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{3n}\left[(-1)^n - \cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3} + 1\right]\sin nx$$

$$= \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{4}\sin 8x + \frac{1}{5}\sin 10x + \frac{1}{7}\sin 14x + \cdots \quad \text{——(1)}$$

Now, $f(2\pi/3) = \dfrac{f(2\pi/3 - 0) + f(2\pi/3 + 0)}{2} = -\pi/6$

Putting $x = \pi$ in (1) we have,

$$f(\pi) = 0$$