Reduction Formulae

The formula in which heigher order integral is connected by lower order integral, is called reduction formula.

§ Reduction formula for $\int \sin^n x\,dx$. ($n \in \mathbb{N}, n > 1$)

Let,

$$I_n = \int \sin^n x\,dx.$$

$$\therefore\ I_n = \int \underbrace{\sin^{n-1} x}_{u} \cdot \underbrace{\sin x}_{v}\,dx$$

$$= \sin^{n-1} x\,(-\cos x) - \int \{(n-1)\sin^{n-2} x \cdot \cos x\}(-\cos x)\,dx$$

[Integration by parts]

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= \quad " \quad + (n-1)\int (\sin^{n-2} x - \sin^n x)\,dx$$

$$= \quad " \quad + (n-1)\left[\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right]$$

$$= -\sin^{n-1} x \cos x + (n-1)\left[I_{n-2} - I_n\right]$$

or, $I_n + (n-1)I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

or, $nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

or, $\boxed{I_n = -\frac{1}{n}\sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \cdot I_{n-2}}$ ——— (I)

§ Reduction Formula for $\int_0^{\pi/2} \sin^n x\,dx$. ($n \in \mathbb{N}, n > 1$)

Let, $J_n = \int_0^{\pi/2} \sin^n x\,dx$.

$$\therefore\ J_n = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx$$

or, $J_n = \left[\sin^{n-1}x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2}\{(n-1)\sin^{n-2}x\cdot\cos x\}(-\cos x)\,dx$

$= 0 + (n-1)\int_0^{\pi/2}\sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= (n-1)\left(\int_0^{\pi/2}\sin^{n-2}x\,dx - \int_0^{\pi/2}\sin^n x\,dx\right)$.

$= (n-1)\left(J_{n-2} - J_n\right)$

or, $J_n + (n-1)J_n = (n-1)J_{n-2}$

or, $nJ_n = (n-1)J_{n-2}$

or, $\boxed{J_n = \frac{n-1}{n}\cdot J_{n-2}}$ ---- (II)

Alternative process

$$J_n = \int_0^{\pi/2}\sin^n x\,dx = \left[I_n\right]_0^{\pi/2}$$

$$= \left[-\frac{1}{n}\sin^{n-1}x\cdot\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\left[I_{n-2}\right]_0^{\pi/2}, \text{ by (I)}$$

$$= 0 + \frac{n-1}{n}J_{n-2}$$

i.e, $\boxed{J_n = \frac{n-1}{n}J_{n-2}}$ ---- (II)

[For this alternative process you have to establish the reduction formula for $\int\sin^n x\,dx$, first]

Note 1. We have

$$J_n = \frac{n-1}{n}J_{n-2}$$

$$= \frac{n-1}{n}\cdot\left(\frac{n-3}{n-2}J_{n-4}\right)$$

[Replacing n by $n-2$ in (II), $J_{n-2} = \frac{n-3}{n-2}J_{n-4}$]

or, $J_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} J_{n-4}$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \left(\frac{n-5}{n-4} \cdot J_{n-6}\right)$$

[Replacing n by (n-4) in (II) $J_{n-4} = \frac{n-5}{n-4} J_{n-6}$]

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot J_{n-6}$$

$$= \ldots$$

$$= \ldots$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot J_0 \quad \text{(when n even)}$$

$$\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot J_1 \quad \text{(when n odd)}$$

Now, $J_0 = \int_0^{\pi/2} (\sin x)^0 dx = \int_0^{\pi/2} dx = \pi/2$,

$J_1 = \int_0^{\pi/2} \sin x \, dx = 1.$

$$\therefore \boxed{J_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & (n = \text{even}) \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 & (n = \text{odd}) \end{cases}} \quad \text{--- (III)}$$

§ Reduction formula for $\int \cos^n x \, dx$, $(n \in \mathbb{N}, n > 1)$

Let $A_n = \int \cos^n x \, dx$

$\therefore A_n = \int \underbrace{\cos^{n-1} x}_{u} \cdot \underbrace{\cos x}_{v} \, dx$

$= \cos^{n-1} x \cdot \sin x - \int \{(n-1) \cos^{n-2} x (-\sin x)\} \cdot \sin x \, dx$

[Integration by parts]

or, $A_n = \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$

$= \text{,,} + (n-1)\left[\int \cos^{n-2}x\,dx - \int \cos^n x\,dx\right]$

$= \cos^{n-1}x\sin x + (n-1)\left[A_{n-2} - A_n\right]$

$= \cos^{n-1}x \cdot \sin x + (n-1)A_{n-2} - (n-1)A_n.$

or, $A_n + (n-1)A_n = \cos^{n-1}x \cdot \sin x + (n-1)A_{n-2}$

or, $nA_n = \text{,,}$

or, $\boxed{A_n = \frac{1}{n}\cos^{n-1}x \cdot \sin x + \frac{n-1}{n} \cdot A_{n-2}}$ ---- (IV)

§ Reduction Formula for $\int_0^{\pi/2} \cos^n x\,dx$ $(n \in \mathbb{N}, n > 1)$

Let $B_n = \int_0^{\pi/2} \cos^n x\,dx.$

$\therefore B_n = \int_0^{\pi/2} \cos^{n-1}x \cdot \cos x\,dx$

$= \left[\cos^{n-1}x \cdot \sin x\right]_0^{\pi/2} - \int_0^{\pi/2}\{(n-1)\cos^{n-2}x(-\sin x)\}\sin x\,dx$

$= 0 + (n-1)\int_0^{\pi/2} \cos^{n-2}x(1-\cos^2 x)\,dx$

$= (n-1)\left[\int_0^{\pi/2} \cos^{n-2}x\,dx - \int_0^{\pi/2} \cos^n x\,dx\right]$

$= (n-1)(B_{n-2} - B_n)$

or, $B_n + (n-1)B_n = (n-1)B_{n-2}$

or, $nB_n = (n-1)B_{n-2}$

or, $\boxed{B_n = \frac{n-1}{n} B_{n-2}}$ ---- (V)

i.e $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}x\,dx$

Alternative Process - 1

$$B_n = \int_0^{\pi/2} \cos^n x\,dx = \Big[A_n\Big]_0^{\pi/2}$$

$$= \left[\frac{1}{n}\cos^{n-1}x\cdot\sin x\right]_0^{\pi/2} + \frac{n-1}{n}\Big[A_{n-2}\Big]_0^{\pi/2}, \text{ by (IV)}$$

$$= \frac{n-1}{n} B_{n-2}$$

[For this process you must establish the relation (IV) first]

~~Alternative Process-2.~~

Note-2.

$$B_n = \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^n(\pi/2 - x)\,dx = \int_0^{\pi/2} \sin^n x\,dx = J_n.$$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-4}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} & (n = \text{even}) \\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-4}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}\cdot 1 & (n = \text{odd}) \end{cases}$$

by (III)

~~§ Reduction formula for $\int \tan^n x\,dx$~~.

Note-3

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n\left(\tfrac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = B_n.$$

Thus, one can establish the reduction formula for J_n i.e $\int_0^{\pi/2} \sin^n x\,dx$ from the reduction formula for B_n i.e $\int_0^{\pi/2} \cos^n x\,dx$ and vice versa.

Infact,

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = B_n = \frac{n-1}{n} B_{n-2}, \text{ by (V)}$$

$$= \frac{n-1}{n} \cdot J_{n-2}$$

$$\left[\because B_n = J_n \Rightarrow B_{n-2} = J_{n-2}\right]$$

§ Reduction formula for $\int \tan^n x\,dx$ $(n>1, n\in\mathbb{N})$

Let,

$$T_n = \int \tan^n x\,dx.$$

$$\therefore T_n = \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \int z^{n-2}\,dz - T_{n-2}, \quad \text{where } z = \tan x, \therefore dz = \sec^2 x\,dx.$$

$$= \frac{z^{n-1}}{n-1} - T_{n-2}$$

$$\text{or, } \boxed{T_n = \frac{\tan^{n-1} x}{n-1} - T_{n-2}} \quad \text{--- (VI)}$$

$$\text{i.e. } \boxed{\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx}$$

§ Reduction Formula for $\int_0^{\pi/4} \tan^n x\,dx$ $(n>1, n\in\mathbb{N})$

$$\text{Let } U_n = \int_0^{\pi/4} \tan^n x\,dx = [T_n]_0^{\pi/4}$$

$$\therefore U_n = \left[\frac{\tan^{n-1}}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$\text{or, } \boxed{U_n = \frac{1}{n-1} - U_{n-2}} \quad \text{--- (VII)}$$

§ Reduction formula for $\int \sec^n x\,dx$, $(n \in \mathbb{N}, n>1)$

Let $E_n = \int \sec^n x\,dx$.

$\therefore E_n = \int \underbrace{\sec^{n-2}x}_{u}\, \underbrace{\sec^2 x}_{v}\,dx$

$= \sec^{n-2}x \cdot \tan x - \int \{(n-2)\sec^{n-3}x \cdot \sec x \tan x\} \cdot \tan x\,dx$

[Integration by Parts]

$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n-2}x \cdot \tan^2 x\,dx$

$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n-2}x \cdot (\sec^2 x - 1)\,dx$.

$=$,, $-(n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx$

or, $E_n = \sec^{n-2}x \tan x - (n-2)E_n + (n-2)E_{n-2}$

or, $E_n + (n-2)E_n = \sec^{n-2}x \tan x + (n-2)E_{n-2}$

or, $(n-1)E_n = \sec^{n-2}x \cdot \tan x + (n-2)E_{n-2}$

or, $\boxed{E_n = \frac{1}{n-1}\sec^{n-2}x \cdot \tan x + \frac{n-2}{n-1} \cdot E_{n-2}}$

i.e $\boxed{\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx}$ --- (VIII)

§ Reduction formula for $\int_0^{\pi/4} \sec^n x\,dx$, $(n>1, n \in \mathbb{N})$

Let $F_n = \int_0^{\pi/4} \sec^n x\,dx = \left[E_n\right]_0^{\pi/4}$

$\therefore F_n = \left[\frac{1}{n-1}\sec^{n-2}x \cdot \tan x\right]_0^{\pi/4} + \frac{n-2}{n-1}\left[E_{n-2}\right]_0^{\pi/4}$

or $F_n = \frac{1}{n-1}(\sqrt{2})^{n-2} + \frac{n-2}{n-1}F_{n-2}$ --- (IX) i.e $\int_0^{\pi/4} \sec^n x\,dx = \frac{1}{n-1}(\sqrt{2})^{n-2} + \frac{n-2}{n-1}\int_0^{\pi/4} \sec^{n-2} x\,dx$ ...... (IX) (37)

§ Reduction Formula for $\int (\log x)^n dx$, $(n \in \mathbb{N})$.

Let, $L_n = \int (\log x)^n dx$.

$\therefore L_n = \int (\log x)^n \cdot 1\, dx$

$= (\log x)^n \cdot x - \int \left\{ n(\log x)^{n-1} \cdot \frac{1}{x} \right\} \cdot x\, dx$

$= x(\log x)^n - n\int (\log x)^{n-1} dx$ [Integration by parts]

or, $\boxed{L_n = x(\log x)^n - n \cdot L_{n-1}}$

§ Reduction formula for $\int \sin^m x \cdot \cos^n x\, dx$.

Let, $I_{m,n} = \int \sin^m x \cdot \cos^n x\, dx$.

$\therefore I_{m,n} = \int \underbrace{\cos^{n-1} x}_{u} \cdot \underbrace{(\sin^m x \cdot \cos x)}_{v} dx$.

We know that, $\int \sin^m x \cdot \cos x\, dx = \frac{\sin^{m+1} x}{m+1}$. Therefore, from the above integral

$I_{m,n} = \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int \left\{ (n-1)\cos^{n-2} x(-\sin x) \right\} \cdot \frac{\sin^{m+1} x}{m+1} dx$

$= \frac{1}{m+1}\cos^{n-1} x \cdot \sin^{m+1} x + \frac{n-1}{m+1}\int \cos^{n-2} x \cdot \sin^m x(1-\cos^2 x)\, dx$

$= \text{,,} + \frac{n-1}{m+1}\int \sin^m x \cdot \cos^{n-2} x\, dx - \frac{n-1}{m+1}\int \sin^m x \cdot \cos^n x\, dx$

or, $I_{m,n} = \frac{1}{m+1}\cos^{n-1} x \cdot \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$

or, $I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{1}{m+1} \cos^{n-1}x \cdot \sin^{m+1}x + \frac{n-1}{m+1} I_{m,n-2}$

or, $\frac{m+n}{m+1} I_{m,n} = $ "

or, $\boxed{I_{m,n} = \frac{1}{m+n} \cos^{n-1}x \cdot \sin^{m+1}x + \frac{n-1}{m+n} I_{m,n-2}}$ --- (X)

Similarly, considering

$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \sin^{m-1}x \cdot (\cos^n x \cdot \sin x) dx$.

and noting, $\int \cos^n x \cdot \sin x \, dx = -\frac{\cos^{n+1}x}{n+1}$, one can show that

$\boxed{I_{m,n} = -\frac{1}{m+n} \sin^{m-1}x \cdot \cos^{n+1}x + \frac{m-1}{m+n} I_{m-2,n}.}$ --- (XI)

<u>Remark</u> - If anyone want to decrease the value of 'n' i.e. the power of $\cos x$, use reduction formula (X). And, to decrease the value of 'm' i.e. the power of $\sin x$, use reduction formula (XI).

§ Reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

Let, $J_{m,n} = \int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = [I_{m,n}]_0^{\frac{\pi}{2}}$

$\therefore J_{m,n} = \left[\frac{1}{m+n} \cos^{n-1}x \sin^{m+1}x\right]_0^{\pi/2} + \frac{n-1}{m+n} [I_{m,n-2}]_0^{\frac{\pi}{2}}$, by (X)

$= 0 + \frac{n-1}{m+n} J_{m,n-2}$

or, $\boxed{J_{m,n} = \frac{n-1}{m+n} J_{m,n-2}}$ --- (XII)

Also, from (XI), we have

$$\boxed{J_{m,n} = \frac{m-1}{m+n} J_{m-2,n}} \quad \text{---(XIII)}$$

$$\therefore J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} \text{ or } \frac{m-1}{m+n} J_{m-2,n}$$

Note-4

$$J_{m,n} = \int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \int_0^{\pi/2} \sin^m\left(\frac{\pi}{2}-x\right) \cdot \cos^n\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} \cos^m x \cdot \sin^n x\,dx = J_{n,m}.$$

§ Reduction formula for $\int \cos^m x \cdot \sin nx\,dx$

Let, $I_{m,n} = \int \underset{u}{\cos^m x} \cdot \underset{v}{\sin nx}\,dx$

$$\therefore I_{m,n} = \cos^m x \cdot \left(-\frac{\cos nx}{n}\right) - \int \left\{m \cdot \cos^{m-1} x \cdot (-\sin x)\right\} \cdot \left(-\frac{\cos nx}{n}\right) dx$$

$$= -\frac{1}{n} \cdot \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \cdot (\sin x \cdot \cos nx)\,dx.$$

Since, $\sin(n-1)x = \sin(nx-x) = \sin nx \cdot \cos x - \cos nx \cdot \sin x$, we have

$$\sin x \cdot \cos nx = \sin nx \cdot \cos x - \sin(n-1)x.$$

Using this,

$$I_{m,n} = -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \left\{\sin nx \cos x - \sin(n-1)x\right\} dx$$

$$= \text{''} - \frac{m}{n} \int \cos^m x \cdot \sin nx\,dx + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x\,dx.$$

$$= \text{''} - \frac{m}{n} \cdot I_{m,n} + \frac{m}{n} I_{m-1,n-1}.$$

or, $I_{m,n} + \frac{m}{n} I_{m,n} = -\frac{1}{n} \cos^m x \cdot \cos nx + \frac{m}{n} I_{m-1,n-1}.$

or, $\frac{m+n}{n} I_{m,n} = $,,

or, $\boxed{I_{m,n} = -\frac{1}{m+n}\cos^m x \cdot \cos nx + \frac{m}{m+n} I_{m-1,n-1}.}$

§ Reduction formula for $\int \cos^m x \cdot \cos nx\, dx$

Let $I_{m,n} = \int \underbrace{\cos^m x}_{u} \cdot \underbrace{\cos nx}_{v}\, dx$.

$\therefore I_{m,n} = \cos^m x \cdot \frac{\sin nx}{n} - \int \left\{ m\cos^{m-1} x \cdot (-\sin x) \right\} \frac{\sin nx}{n}\, dx$

$= \frac{1}{n}\cos^m x \cdot \sin nx + \frac{m}{n}\int \cos^{m-1} x \cdot (\sin x \cdot \sin nx)\, dx$.

From,

$\cos(n-1)x = \cos(nx - x) = \cos nx \cdot \cos x + \sin nx \cdot \sin x$,

where

$\sin x \cdot \sin nx = \cos(n-1)x - \cos nx \cdot \cos x$.

$\therefore I_{m,n} = \frac{1}{n}\cos^m x \cdot \sin nx + \frac{m}{n}\int \cos^{m-1} x \left\{ \cos(n-1)x - \cos nx \cos x \right\} dx$

$=$,, $+ \frac{m}{n}\int \cos^{m-1} x \cdot \cos(n-1)x\, dx - \frac{m}{n}\int \cos^m x \cos nx\, dx$

$= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$.

or, $I_{m,n} + \frac{m}{n} I_{m,n} = \frac{1}{n}\cos^m x \cdot \sin nx + \frac{m}{n} I_{m-1,n-1}$

or, $I_{m,n} = \frac{1}{m+n}\cos^m x \cdot \sin nx + \frac{m}{m+n} I_{m-1,n-1}$.

Example

1. Find a reduction formula for $\int \sin^n x\,dx$, where $n>1$ and $n \in \mathbb{N}$. Hence find the value of $\int \sin^6 x\,dx$.

Solⁿ– $\underline{1^{st}\text{ part}}$ – See page no- 30, equation (I).

$\underline{2^{nd}\text{ part}}$ – We have

$$I_n = -\frac{1}{n}\sin^{n-1}x\cdot\cos x + \frac{n-1}{n}\cdot I_{n-2} \qquad -(1)$$

Putting $n=6$ in (1), we have

$$I_6 = -\frac{1}{6}\sin^5 x\cdot\cos x + \frac{5}{6}I_4$$

$$= -\frac{1}{6}\sin^5 x\cdot\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cdot\cos x + \frac{3}{4}I_2\right), \text{ by putting } n=4 \text{ in } (1)$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6}\cdot\left(-\frac{1}{4}\sin^3 x\cdot\cos x\right) + \frac{5}{6}\cdot\frac{3}{4}\cdot I_2$$

$$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0\right),$$

by putting $n=2$ in (1)

$$= -\frac{1}{6}\sin^5 x\cdot\cos x + \frac{5}{24}\sin^3 x\cdot\cos x - \frac{5}{16}\sin x\cdot\cos x + \frac{5}{16}x + c,$$

(c = arbitrary constant)

since, $I_0 = \int \sin^0 x\,dx = x$.

Ex-2 Find a reduction formula for $\int \cos^n x\,dx$ where $n>1$ and $n \in \mathbb{N}$. Also evaluate $\int \cos^5 x\,dx$ and $\int \cos^6 x\,dx$.

Sol^n – Home Work.

Ex-3. Deduce the reduction formula for $\int_0^{\pi/2} \sin^n \theta\, d\theta$ where $n>1$ and $n \in \mathbb{N}$. Hence, find the value of $\int_0^{\pi/2} \sin^7\theta\, d\theta$.

Sol^n – let,
$$J_n = \int_0^{\pi/2} \sin^n\theta\, d\theta.$$

For next step see page-30, establish equation (II).

$$J_n = \frac{n-1}{n} J_{n-2} \quad \text{---} (1)$$

Putting $n=7$ in (i), we obtain

$$J_7 = \frac{6}{7}\cdot J_5$$
$$= \frac{6}{7}\cdot\left(\frac{4}{5} J_3\right), \text{ by putting } n=5 \text{ in } (1)$$
$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\left(\frac{2}{3}\cdot J_1\right), \text{ " " } n=3 \text{ " } (i)$$
$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \text{ since, } J_1 = \int_0^{\pi/2} \sin x\, dx = 1$$
$$= 16/35.$$

Ex-4. If n is even positive integer, then show that $\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2$. Hence find $\int_0^{\pi/2}\cos^{10}\theta\, d\theta$.

Hints – First establish relation (III) (see page nos- 31, 32) for even case. After that, see Note-2 (page-34).

Ex-5 If $I_n = \int_0^{\pi/2} \sin^n x\,dx$, where n is a positive integer greater than 1, prove that $I_n = \frac{n-1}{n} I_{n-2}$.
Hence show that,

(a) $\int_0^{\pi/2} \sin^8 x\,dx = \frac{35\pi}{256}$ and (b) $\int_0^1 \frac{x^6\,dx}{\sqrt{1-x^2}} = \frac{5\pi}{32}$.

Hints - (b) Put $x = \sin\theta$

$$\int_0^1 \frac{x^6\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\sin^6\theta \cdot \cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \int_0^{\pi/2} \sin^6\theta\,d\theta = I_6$$

Next use reduction formula.

Ex-6. If $J_n = \int_0^{\pi/2} \cos^n\theta\,d\theta$, $(n>1, n \in \mathbb{N})$, show that $J_n = \frac{n-1}{n} J_{n-2}$. Hence, find $\int_0^{\pi/2} \cos^4\theta \sin^2\theta\,d\theta$.

Sol^n - 1st part - page-33, relation (V).

2nd part - $\int_0^{\pi/2} \cos^4\theta \sin^2\theta\,d\theta$

$$= \int_0^{\pi/2} \cos^4\theta(1-\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi/2} \cos^4\theta\,d\theta - \int_0^{\pi/2} \cos^6\theta\,d\theta$$

$$= J_4 - J_6$$

$$= \frac{3}{4} J_2 - \frac{5}{6} J_4$$

$$= \frac{3}{4} \cdot \frac{1}{2} J_0 - \frac{5}{6} \cdot \frac{3}{4} \cdot J_2$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} J_0 \qquad \left[\because J_0 = \int_0^{\pi/2} \cos^0\theta\,d\theta = \pi/2\right]$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \pi/2 - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi/2$$

$$= \pi/32$$

Ex-7 If $I_n = \int_0^{\pi/2} \sin^{2n+1} x\, dx$, $(n \in \mathbb{N})$, show that $I_n = \frac{2n}{2n+1} I_{n-1}$. Hence, show that $\int_0^{\pi/2} \sin^5 x\, dx = 8/15$

Soln – Given,

$$I_n = \int_0^{\pi/2} \sin^{2n+1} x\, dx \qquad ---(1)$$

$$= \int_0^{\pi/2} \sin^{2n} x \cdot \sin x\, dx$$

$$= \left[-\sin^{2n} x \cdot \cos x\right]_0^{\pi/2} - \int \left(2n \sin^{2n-1} x \cdot \cos x\right)(-\cos x)\, dx$$

[Integration by parts]

$$= 0 + 2n \int_0^{\pi/2} \sin^{2n-1} x \cdot \cos^2 x\, dx$$

$$= 2n \int_0^{\pi/2} \sin^{2n-1} x\,(1 - \sin^2 x)\, dx$$

or $I_n = 2n \left[\int_0^{\pi/2} \sin^{2n-1} x\, dx - \int_0^{\pi/2} \sin^{2n+1} x\, dx \right]$

or $I_n = 2n \left[\int_0^{\pi/2} \sin^{2(n-1)+1} x\, dx - I_n \right]$

or $I_n = 2n \left(I_{n-1} - I_n \right)$

or, $I_n + 2n I_n = 2n I_{n-1}$

or, $I_n = \frac{2n}{1+2n} I_{n-1}$ (Proved). $---(2)$

Substituting $n = 2$ in (1), we get

$$I_2 = \int_0^{\pi/2} \sin^5 x\, dx$$

$$= \frac{4}{5} I_1, \text{ by } (2)$$

$$= \frac{4}{5} \cdot \frac{2}{3} \cdot I_0, \text{ by } (2)$$

$$= \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = 8/15, \text{ since } I_0 = \int_0^{\pi/2} \sin x\, dx = 1$$

Ex-8 Obtain a reduction formula for $\int \tan^n x\,dx$ and hence find the values of $\int \tan^5 x\,dx$ and $\int \tan^6 x\,dx$.

Sol^n – First part – Page-35, equation (VI)

Second part –

We have, $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$. ——(1)

$\therefore I_5 = \int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - I_3$, by (1)

$= \frac{1}{4}\tan^4 x - \left(\frac{1}{2}\tan^2 x - I_1\right)$, by putting $n=3$ in (1)

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log\sec x + c$, ($c$ = arbitrary const.)

Since $I_1 = \int \tan x\,dx = \log\sec x + c$.

$\int \tan^6 x\,dx \longrightarrow$ H.W.

Ex-9 Find a reduction formula for $\int \sec^n x\,dx$. Hence find the values of $\int \sec^7 x\,dx$ & $\int \sec^6 x\,dx$.

Sol^n – H.W.

Ex-10 If $I_n = \int \sec^n x\,dx$, show that

$$(n-1)I_n = \tan x \cdot \sec^{n-2} x + (n-2)I_{n-2}.$$

Hence, find a reduction formula for $\int_0^{\pi/4} \sec^n x\,dx$. Use this formula find the value of $\int_0^{\pi/4} \sec^5 x\,dx$.

Sol^n - 1st part - Page No - 36, equation (VIII)

2nd part ~~[illegible]~~

Let, $J_n = \int_0^{\pi/4} \sec^n x\,dx = [I_n]_0^{\pi/4}$

$\therefore$ From 1st part

$$(n-1)[I_n]_0^{\pi/4} = [\tan x \cdot \sec^{n-2} x]_0^{\pi/4} + [(n-2) I_{n-2}]_0^{\pi/4}$$

or, $(n-1) J_n = (\sqrt{2})^{n-2} + (n-2) J_{n-2}$

or, $J_n = \frac{1}{n-1}(\sqrt{2})^{n-2} + \frac{n-2}{n-1} J_{n-2}$ --- (1)

which is the reduction formula for $\int_0^{\pi/4} \sec^n x\,dx$.

3rd part -

Putting $n=5$ in (1),

$$J_5 = \frac{1}{4}(\sqrt{2})^3 + \frac{3}{4} J_3.$$

$$= \frac{1}{4}\cdot 2\sqrt{2} + \frac{3}{4}\left[\frac{1}{2}(\sqrt{2})^1 + \frac{1}{2} J_1\right]$$, by putting $n=3$ in (1)

$$= \frac{1}{2}\sqrt{2} + \frac{3}{4}\cdot\frac{1}{2}\sqrt{2} + \frac{3}{4}\cdot\frac{1}{2} J_1$$

Now, $J_1 = \int_0^{\pi/4} \sec x\,dx = [\log(\sec x + \tan x)]_0^{\pi/4}$

$$= \log(\sqrt{2}+1)$$

$\therefore J_5 = \int_0^{\pi/4} \sec^5 x\,dx = \frac{1}{2}\sqrt{2} + \frac{3}{8}\sqrt{2} + \frac{3}{8}\log(\sqrt{2}+1)$

$$= \frac{7}{8}\sqrt{2} + \frac{3}{8}\log(\sqrt{2}+1).$$

Ex - 11 If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, show that

$$I_{n+1} + I_{n-1} = \frac{1}{n}.$$

Use ~~that~~ this formula to evaluate $\int_0^{\pi/4} \tan^7 x\,dx$.

Soln - 1st part - page-35, from (VII), we got

$$I_n = \frac{1}{n-1} - I_{n-2} \quad \text{---(i)}$$

Replacing n by $n+1$ in (i), we get

$$I_{n+1} = \frac{1}{(n+1)-1} - I_{(n+1)-2}$$

$$\text{or, } I_{n+1} = \frac{1}{n} - I_{n-1}$$

$$\text{or, } I_{n+1} + I_{n-1} = \frac{1}{n} \quad \text{(Proved)}$$

2nd part -

Substituting $n = 7, 5, 3$ successively in (i), we have

$$I_7 = \frac{1}{6} - I_5,$$

$$I_5 = \frac{1}{4} - I_3,$$

$$\text{and } I_3 = \frac{1}{2} - I_1$$

Then,

$$I_7 = \frac{1}{6} - \left(\frac{1}{4} - I_3\right) = \frac{1}{6} - \frac{1}{4} + I_3$$

$$= \frac{1}{6} - \frac{1}{4} + \left(\frac{1}{2} - I_1\right)$$

$$= \frac{5}{12} - \log\sqrt{2},$$

since, $I_1 = \int_0^{\pi/4} \tan x\,dx = [\log \sec x]_0^{\pi/4} = \log\sqrt{2}$, Ex-12 Find a reduction formula for $\int \sin^m x \cos^n x\, dx$, where m and n are positive integers. Hence, find a reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$. Use the above formula to evaluate $\int_0^{\pi/2} \sin^8 x \cdot \cos^6 x\, dx$.

Solⁿ- 1st part - page No- 37, 38, equation (X) or (XI)

2nd Part - page- 38, 39, equation (XII) or (XIII)

3rd part - $\boxed{\text{suppose you establish (XIII)}}$

$$J_{m,n} = \frac{m-1}{m+n} J_{m-2,n} \quad \text{---(1)}$$

Here, $J_{8,6} = \int_0^{\pi/2} \sin^8 x \cdot \cos^6 x\, dx$

$= \frac{7}{14} \cdot J_{6,6}$, by putting $m=8, n=6$ in (1)

$= \frac{7}{14} \cdot \frac{5}{12} J_{4,6}$ " " $m=6, n=6$ in (1)

$= \frac{7}{14} \cdot \frac{5}{12} \cdot \frac{3}{10} J_{2,6}$ " " $(m=4, n=6$ " "

$= \frac{7}{14} \cdot \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} J_{0,6}$ " " $m=2, n=6$ " "

$= \frac{7}{14} \cdot \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot \int_0^{\pi/2} \cos^6 x\, dx$

$= \frac{7}{14} \cdot \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot \left(\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)$,

$= 5\pi/4096$,

since, $\int_0^{\pi/2} \cos^6 x\, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ (See Note-2, page No - 34)

$\boxed{\text{Alternative process}}$

$J_{8,6} = \text{-----} = \frac{7}{14} \cdot \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot \int_0^{\pi/2} \cos^6 x\, dx$

$\cancel{\therefore J_{8,6} = \frac{7}{14}\cdot\frac{5}{12}\cdot\frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\int_0^{\pi/2}\cos^4 x\,dx)}$ ~~(By the formula)~~

Now, $\displaystyle\int_0^{\pi/2}\cos^6 x\,dx = \frac{6-1}{6}\int_0^{\pi/2}\cos^4 x\,dx$ (formula-(V), page no-33)

$$= \frac{5}{6}\cdot\frac{3}{4}\int_0^{\pi/2}\cos^2 x\,dx$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2}(\cos x)^0\,dx$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2$$

$$\therefore\ J_{8,6} = \frac{7}{14}\cdot\frac{5}{12}\cdot\frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= 5\pi/4096.$$

Ex-13 If $I_{m,n} = \int \cos^m x\cdot\sin^n x\,dx$, then show that

$$(m+n)(m+n-2)I_{m,n} = \{(n-1)\sin^2 x - (m-1)\cos^2 x\}\cdot\cos^{m-1}x\sin^{n-1}x + (m-1)(n-1)I_{m-2,n-2}$$

Hints -

$$I_{m,n} = \frac{1}{m+n}\sin^{n+1}x\cdot\cos^{m-1}x + \frac{m-1}{m+n}\cdot I_{m-2,n} \quad \text{---(1)}$$

$$= -\frac{1}{m+n}\sin^{n-1}x\cos^{m+1}x + \frac{n-1}{m+n}\cdot I_{m,n-2} \quad \text{---(2)}$$

Replacing m by $m-2$ in (2), we get

$$I_{m-2,n} = -\frac{1}{m+n-2}\cdot\sin^{n-1}x\cdot\cos^{m-1}x + \frac{n-1}{m+n-2}\cdot I_{m-2,n-2}$$

From (1),

$$(m+n)\, I_{m,n} = \sin^{n+1}x \cdot \cos^{m-1}x + (m-1)\, I_{m-2,n}$$

$$= \sin^{m+1}x \cdot \cos^{m-1}x$$

$$+ (m-1)\left[-\frac{1}{m+n-2}\sin^{n-1}x\cos^{m-1}x + \frac{n-1}{m+n-2}\cdot I_{m-2,n-2}\right],$$

by (3)

or, $$(m+n)(m+n-2)\, I_{m,n} = (m+n-2)\sin^{n+1}x\cos^{m-1}x$$

$$- (m-1)\sin^{n-1}x\cos^{m-1}x + (m-1)(n-1)\, I_{m-2,n-2}$$

$$= \{(m+n-2)\sin^2 x - (m-1)\}\sin^{n-1}x\cdot\cos^{m-1}x$$

$$+ (m-1)(n-1)\, I_{m-2,n-2}.$$

$$\cancel{= \{(n-1)\sin^2 x + (m-1)\sin^2 x - (m-1)\}\sin^{n-1}x\cos^{m-1}x + (m-1)(n-1)\, I_{m-2,n-2}}$$

$$\cancel{= \{(n-1)\sin^2 x - (m-1)(1-\sin^2 x)\}\sin^{n-1}x\cos^{m-1}x + (m-1)(n-1)\, I_{m-2,n-2}}$$

$$= \{(m-1+m-1)\sin^2 x - (m-1)\}\sin^{n-1}x\cdot\cos^{m-1}x$$

$$+ (m-1)(n-1)\, I_{m-2,n-2}$$

$$= \{(n-1)\sin^2 x + (m-1)\sin^2 x - (m-1)\}\sin^{n-1}x\cos^{m-1}x$$

$$+ (m-1)(n-1)\, I_{m-2,n-2}$$

$$= \{(n-1)\sin^2 x - (m-1)(1-\sin^2 x)\}\cdot\cos^{m-1}x\sin^{n-1}x$$

$$+ (m-1)(n-1)\, I_{m-2,n-2}$$

$$= \{(n-1)\sin^2 x - (m-1)\cos^2 x\}\cos^{m-1}x\sin^{\cancel{m-1}}x$$

$$+ (m-1)(n-1)\, I_{m-2,n-2} \quad \text{(Proved)}$$

$$= x^{m+1}\left[\frac{(\log x)^n}{m+1}-\frac{n(\log x)^{n-1}}{(m+1)^2}+\frac{n(n-1)(\log x)^{n-2}}{(m+1)^3}-\text{.......}+\frac{(-1)^n n!}{(m+1)^{n+1}}\right]$$

Example 18. Find the reduction formula for $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$, m, n being positive integers and hence deduce that

$$I_{m,\,m} = \frac{1}{2^{m+1}}\left[2+\frac{2^2}{2}+\frac{2^3}{3}+\text{.......}+\frac{2^m}{m}\right].$$

Solution : Given $I_{m,\,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$

$$= \left[-\frac{\cos^m x \cos nx}{n}\right]_0^{\frac{\pi}{2}} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin x \cos nx\, dx$$

$$= \frac{1}{n}-\frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x\{\sin nx \cos x - \sin(n-1)x\}\, dx$$

$$[\because \sin(n-1)x = \sin nx \cos x - \cos nx \sin x]$$

$$= \frac{1}{n}-\frac{m}{n}I_{m,n}+\frac{m}{n}I_{m-1,n-1}$$

$$\Rightarrow \left(1+\frac{m}{n}\right)I_{m,n} = \frac{1}{n}+\frac{m}{n}I_{m-1,n-1}$$

$$\Rightarrow (m+n)I_{m,n} = 1+mI_{m-1,n-1} \quad \Rightarrow I_{m,n} = \frac{1}{m+n}+\frac{m}{m+n}\cdot I_{m-1,n-1}$$

This is the required reduction formula.

Now, $I_{m,\,m} = \frac{1}{2m}+\frac{1}{2}I_{m-1,m-1}$

$$= \frac{1}{2m}+\frac{1}{2}\left\{\frac{1}{2(m-1)}+\frac{1}{2}I_{m-2,m-2}\right\}$$

$$= \frac{1}{2m}+\frac{1}{2^2(m-1)}+\frac{1}{2^2}\left\{\frac{1}{2(m-2)}+\frac{1}{2}I_{m-3,m-3}\right\}$$

$$= \frac{1}{2m}+\frac{1}{2^2(m-1)}+\frac{1}{2^3(m-2)}+\frac{1}{2^3}I_{m-3,m-3}$$

$$= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \frac{1}{2m}+\frac{1}{2^2(m-1)}+\frac{1}{2^3(m-2)}+\ldots+\frac{1}{2^{m-2}.3}+\frac{1}{2^{m-1}.2}+\frac{1}{2^{m-1}}I_{1,1}$$

Since, $I_{1,\,1} = \int_0^{\frac{\pi}{2}} \cos x \sin x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$\therefore I_{m,\,m} = \frac{1}{2m}+\frac{1}{2^2(m-2)}+\frac{1}{2^3(m-3)}+\ldots+\frac{1}{2^{m-2}.3}+\frac{1}{2^{m-1}.2}+\frac{1}{2^m}$$

$$= \frac{1}{2^{m+1}}\left[2+\frac{2^2}{2}+\frac{2^3}{3}+\text{......}+\frac{2^m}{m}\right].$$

$$= \frac{1}{2}\int \frac{\sec^2 \frac{x}{2} dx}{4-\tan^2 \frac{x}{2}} \quad \text{Put } \tan\frac{x}{2}=z \therefore \frac{1}{2}\sec^2 \frac{x}{2} dx = dz$$

$$= \int \frac{dz}{4-z^2}$$

$$= \frac{1}{4}\log\left|\frac{2+z}{2-z}\right|$$

$$= \frac{1}{4}\log\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| \qquad \text{... (2)}$$

$\therefore$ From (1) & (2) we write,

$$I_2 = \frac{5}{16}\cdot\frac{\sin x}{3+5\cos x} - \frac{3}{64}\log\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + c.$$

Example 21. If $I_n = \int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^n dx$, then show that $nI_n = ab(a^{n-2} + b^{n-2}) + (n-1)(a^2+b^2) I_{n-2}$, where n is a positive integer greater than or equal to 2.

» Solution : Given $I_n = \int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^n dx$

$$= \int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-1}.(a\cos x + b\sin x)dx$$

$$= \left[(a\cos x + b\sin x)^{n-1}(a\sin x - b\cos x)\right]_0^{\frac{\pi}{2}}$$

$$-(n-1)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2}.(-a\sin x + b\cos x).(a\sin x - b\cos x)dx$$

[Integration by parts taking $(a \cos n + b \sin x)^n$ as first function and $(a \cos x + b \sin x)$ as 2nd function]

$$= \left(ab^{n-1} + a^{n-1}b\right) + (n-1)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2}.(a\sin x - b\cos x)^2 dx$$

$$= ab\left(a^{n-2} + b^{n-2}\right) + (n-1)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2}.$$
$$\left(a^2\sin^2 x + b^2\cos^2 x - 2ab\sin x\cos x\right)dx$$

$$= ab\left(a^{n-2} + b^{n-2}\right) + (n-1)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2}.$$
$$\left[a^2 - a^2\cos^2 x + b^2 - b^2\sin^2 x - 2ab\sin x\cos x\right]dx$$

$$= ab\left(a^{n-2} + b^{n-2}\right) + (n-1)\left(a^2+b^2\right)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2} dx$$
$$-(n-1)\int_0^{\frac{\pi}{2}} (a\cos x + b\sin x)^{n-2}.(a\cos x + b\sin x)^2 dx$$

$$= ab\left(a^{n-2} + b^{n-2}\right) + (n-1)\left(a^2+b^2\right)I_{n-2} - (n-1)I_n$$

$$\Rightarrow (1+n-1)I_n = ab\left(a^{n-2} + b^{n-2}\right) + (n-1)\left(a^2+b^2\right)I_{n-2}$$

$$\Rightarrow nI_n = ab(a^{n-2} + b^{n-2}) + (n-1)(a^2+b^2)I_{n-2}$$

Now $I_3 = \int x^3 e^{ax} dx$.

Putting $n = 3, 2, 1$ successively in (1) we get

$$I_3 = x^3 \frac{e^{ax}}{a} - \frac{3}{a} I_2$$

$$I_2 = x^2 \frac{e^{ax}}{a} - \frac{2}{a} I_1$$

$$I_1 = x \frac{e^{ax}}{a} - \frac{1}{a} I_0$$

$$I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{e^{ax}}{a}.$$

Hence, $I_3 = x^3 \frac{e^{ax}}{a} - \frac{3}{a}\left\{x^2 \frac{e^{ax}}{a} - \frac{2}{a} I_1\right\}$

or, $I_3 = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2} I_1 = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\left(x\frac{e^{ax}}{a} - \frac{1}{a} I_0\right)$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x e^{ax}}{a^3} - \frac{6e^{ax}}{a^4} + C.$$

Example. 6. $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx$ $(n \geq 1)$ show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

[C.P.1998, 2003]

Hence evaluate $\int_0^{\frac{\pi}{2}} x^4 \sin x dx$.

Solution. Doing integration by parts we get

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx = n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx$$

$$= n\left\{\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)x^{n-2}\sin x dx\right\}$$

$$= n\left\{\left(\frac{\pi}{2}\right)^{n-1} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x dx\right\} = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x dx$$

or, $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$

or, $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ ... (1)

Now, $I_4 = \int_0^{\frac{\pi}{2}} x^4 \sin x dx$

Putting $n = 4, 2$ in (1) we get

$I_4 + 12I_2 = 4\left(\frac{\pi}{2}\right)^3$ or, $I_2 + 2I_0 = 2\cdot\left(\frac{\pi}{2}\right)^{2-1} = \pi$ or, $I_2 = \pi - 2I_0$

Hence, $I_4 = 4\cdot\frac{\pi^3}{8} - 12I_2 = \frac{\pi^3}{2} - 12\{\pi - 2I_0\} = \frac{\pi^3}{2} - 12\pi + 24I_0$.

Now, $I_0 = \int_0^{\frac{\pi}{2}} x^0 \sin x dx = -\left[\cos x\right]_0^{\frac{\pi}{2}} = 1$.

Hence, $I_4 = \frac{\pi^3}{2} - 12\pi + 24$.

Example. 7. If $U_n = \int (\sin x + \cos x)^n dx$ then show that

$$nU_n = -(\sin x + \cos x)^{n-2}\cos 2x + 2(n-1)\ U_{n-2}.$$

Solution. $U_n = \int (\sin x + \cos x)^n \, dx$

or, $U_n = \int (\sin x + \cos x)^{n-1} (\sin x + \cos x) dx$

Doing integration by parts we get

or, $U_n = (\sin x + \cos x)^{n-1}(-\cos x + \sin x) - \int (n-1)(\sin x + \cos x)^{n-2}(\cos x - \sin x)(-\cos x + \sin x) dx$

$= -(\cos x + \sin x)^{n-2}(\cos^2 x - \sin^2 x) + (n-1)\int (\sin x + \cos x)^{n-2}(\cos x - \sin x)^2$

$U_n = -(\cos x + \sin x)^{n-2} \cos 2x + (n-1)\int (\sin x + \cos x)^{n-2}(1 - 2\sin x \cos x)$

$= -(\cos x + \sin x)^{n-2} \cos 2x + (n-1)\int (\sin x + \cos x)^{n-2} dx - (n-1)\int (\sin x + \cos x)^{n-2} 2\sin x \cos x dx$

$= -(\cos x + \sin x)^{n-2} \cos 2x + (n-1)U_{n-2} - (n-1)\int (\sin x + \cos x)^{n-2}(1 + 2\sin x \cos x - 1) dx$

$= -(\cos x + \sin x)^{n-2} \cos 2x + (n-1)U_{n-2} + (n-1)$

$\int (\sin x + \cos x)^{n-2} dx - (n-1)\int (\sin x + \cos x)^{n-2}(1 + 2\sin x \cos x) dx$

$= -(\cos x + \sin x)^{n-2} \cos 2x + (n-1)U_{n-2} + (n-1)U_{n-2} - (n-1)\int (\sin x + \cos x)^{n-2}(\sin x + \cos x)^2 dx$

or, $U_n = -(\cos x + \sin x)^{n-2} \cos 2x + 2(n-1)U_{n-2} - (n-1)\int (\sin x + \cos x)^n dx$

or, $U_n = -(\cos x + \sin x)^{n-2} \cos 2x + 2(n-1)U_{n-2} - (n-1)U_n$

or, $nU_n = -(\sin x + \cos x)^{n-2} \cos 2x + 2(n-1)U_{n-2}$