Successive Differentiation

I) $y = (ax+b)^m, \; m \in \mathbb{N}$

$\Rightarrow y_n = a^m n!, \quad m = n$

$= 0, \quad m < n$

$= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

II) $y = \log(ax+b)$

$\Rightarrow y_n = (-1)^{n-1} \frac{a^n (n-1)!}{(ax+b)^n}$

III) $y = \frac{1}{ax+b}$

$\Rightarrow y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

IV) $y = \sin(ax+b); \cos(ax+b)$

$\Rightarrow y_n = a^n \sin(n\frac{\pi}{2} + ax+b),$

$a^n \cos(n\frac{\pi}{2} + ax+b).$

V) $y = e^{ax+b}$

$\Rightarrow y_n = a^n e^{ax+b}$

VI) $y = e^{ax}\sin bx$

$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx + n\tan^{-1}\frac{b}{a})$

Problems

Find y_n

1)(i) $y = \frac{1}{a-x} = (a-x)^{-1}$

Sol^n – $y_n = (-1)^n (-1)^n \frac{n!}{(a-x)^{n+1}}$

$= \frac{n!}{(a-x)^{n+1}}$

(ii) $y = \sqrt{x} = x^{1/2}$

Sol^n – $y_n = \frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-\overline{n-1})\, x^{\frac{1}{2}-n}$

$= \frac{1\cdot(-1)(-3)\cdots(-1)(2n-3)}{2^n} x^{\frac{1}{2}-n}$

$= (-1)^{n-1} \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2^n} \cdot x^{\frac{1}{2}-n}$

(iii) $y = x^{-1/2}$

Sol^n - $y_n = (-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-\overline{n-1})\, x^{-\frac{1}{2}-n}$

$= (-1)(-3)(-5)\cdots(-1)(2n-1)\,\dfrac{x^{-\frac{1}{2}-n}}{2^n}$

$= (-1)^n \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n x^{\frac{1}{2}+n}}$

(IV) $y = (2-3x)^n$

Sol^n - $y_n = (-1)^n\, n!\cdot 3^n$

(V) $y = \log(ax+x^2)$

Sol^n - Given, $y = \log(ax+x^2)$

$= \log\{x(a+x)\}$

$= \log x + \log(a+x)$

$\therefore y_n = (-1)^{n-1}(n-1)!\left\{\dfrac{1}{x^n} + \dfrac{1}{(a+x)^n}\right\}$

vi) $y = \dfrac{x^n}{x-1}$

Sol^n - Given,

$y = \dfrac{x^n}{x-1} = \dfrac{x^n-1}{x-1} + \dfrac{1}{x-1}$

$= x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1 + \dfrac{1}{x-1}$

$\therefore y_n = 0 + 0 + \cdots + 0 + 0 + 0 + \dfrac{(-1)^n\, n!}{(x-1)^{n+1}}$

$= (-1)^n n! / (x-1)^{n+1}$

vii) $y = \sin x \sin 2x \cdot \sin 3x$

Sol^n - Given, $y = \sin x \sin 2x \sin 3x$

$= \frac{1}{2}(2\sin x \sin 2x)\sin 3x$

$= \frac{1}{2}(\cos x - \cos 3x)\sin 3x$

$= \frac{1}{2}(\cos x \sin 3x - \cos 3x \sin 3x)$

$= \frac{1}{4}(2\sin 3x \cos x - 2\sin 3x \cos 3x)$

or, $y = \frac{1}{4}(\sin 4x + \sin 2x - \sin 6x)$ (3)

$\therefore\ y_n = \frac{1}{4}\left[4^n \sin\left(\frac{n\pi}{2}+4x\right) + 2^n \sin\left(\frac{n\pi}{2}+2x\right) - 6^n \sin\left(\frac{n\pi}{2}+6x\right)\right]$

(VIII) $y = \dfrac{x^2+1}{(x-1)(x-2)(x-3)}$.

Sol^n Let,

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

$\therefore\ x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

Substituting $x = 1, 2$ and 3 successively on both sides, we get

$A = 1,\ B = -5$ and $C = 5$

$$\therefore\ y = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

Hence, $y_n = (-1)^n n! \left[\frac{1}{(x-1)^{n+1}} - \frac{5}{(x-2)^{n+1}} + \frac{5}{(x-3)^{n+1}}\right]$

(IX) $y = \dfrac{x^2}{(x+1)^2(x+2)}$.

Sol^n - Let,

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$$

Hence, $x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$ --(1)

Putting $x = -1$ and -2 successively on both sides, we get

$B = 1$ and $C = 4$.

Now substituting $x=0$ in (i), we obtain (4)

$$0 = A\cdot 1\cdot 2 + B\cdot 2 + C\cdot 1$$

or, $2A + 2B + C = 0$

or, $2A + 2\cdot 1 + 3 = 0$ $\quad [\because B=1, C=4]$

or, $A = -3$.

$$\therefore y = \frac{-3}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{(x+2)}.$$

Hence,

~~$y_n = (-3)\ (-1)^n n! \left[\frac{-3}{(x+1)^{n+1}} + \right.$~~

$$y_n = \frac{(-3)(-1)^n n!}{(x+1)^{n+1}} + (-1)^n\frac{(n+1)!}{(x+1)^{n+2}} + 4\cdot(-1)^n\frac{n!}{(x+2)^{n+1}}.$$

(X) $y = \dfrac{x^2}{(x+a)(x+b)}$

[Hints:- $y = \dfrac{x^2}{(x+a)(x+b)} = 1 + \dfrac{A}{x+a} + \dfrac{B}{(x-b)}$,

$$A = a^2/(a-b), \quad B = b^2/(b-a)$$

XI) $y = 10^{5-3x}$

Sol^n – $y = 10^{5-3x}$

$= 10^5 \cdot 10^{-3x}$

$= 10^5 \cdot e^{\log 10^{-3x}}$

$= 10^5 e^{-3x\log 10}$

$= 10^5 e^{(-3\log 10)x}$

$= 10^5 e^{ax}$, $a = -3\log 10$.

$\therefore y_n = 10^5 a^n e^{ax}$

$= 10^5 (-3\log 10)^n e^{(-3\log 10)x}$

$= (-1)^n 3^n (\log 10)^n 10^{5-3x}$

2. If $y=(x-1)^n$, then show that.
$$y+\frac{y_1}{1!}+\frac{y_2}{2!}+\cdots+\frac{y_n}{n!}=x^n$$

Sol^n – We have,
$$y=(x-1)^n.$$

$\therefore\ \left[y_1=n(x-1)^{n-1};\ y_2=n(n-1)(x-1)^{n-2}\right],\ y_3=n(n-1)(n-2)x^{n-3},\ \cdots$

$\cdots y_n=n!$

Now,
$$y+\frac{y_1}{1!}+\frac{y_2}{2!}+\cdots+\frac{y_n}{n!}$$
$$=(x-1)^n+\frac{n(x-1)^{n-1}}{1!}+\frac{n(n-1)(x-1)^{n-2}}{2!}+\cdots+\frac{n!}{n!}$$
$$=(x-1)^n+{}^nc_1(x-1)^{n-1}+{}^nc_2(x-1)^{n-2}+\cdots+1$$
$$=\{(x-1)+1\}^n \qquad \left[\because (z+1)^n=z^n+{}^nc_1z^{n-1}+{}^nc_2z^{n-2}+\cdots+1\right]$$
$$=x^n \text{ (Proved)}$$

3. If $y=\frac{x}{x+1}$, show that $y_5(0)=5!$

Sol^n – Here,
$$y=\frac{x}{x+1}=\frac{x+1-1}{x+1}=1-\frac{1}{x+1}$$

$$\therefore\ y_5=(-1)(-1)^5\frac{5!}{(x+1)^6}=\frac{5!}{(x+1)^6}. \qquad \left[\because y_n=(-1)^n\frac{n!}{(x+1)^{n+1}} \text{ for } y=\frac{1}{x+1}\right]$$

Hence
$$y_5(0)=5!$$

4. If $y=2\cos x(\sin x-\cos x)$, show that $y_{10}(0)=2^{10}$.

Sol^n – Given, $y=2\cos x(\sin x-\cos x)=2\cos x\sin x-2\cos^2x$

or, $y=\sin 2x-(\cos 2x+1)=\sin 2x-\cos 2x-1$

$\therefore\ y_{10}=2^{10}\left[\sin\left(10\cdot\frac{\pi}{2}+2x\right)-\cos\left(10\cdot\frac{\pi}{2}+2x\right)\right]$

or $y_{10}(0)=2^{10}\left[\sin 5\pi-\cos 5\pi\right]=2^{10}\left[0-(-1)^5\right]=2^{10}$

~~5. If $y = \cos(10\cos x)$~~

5. If $y = \sin ax + \cos ax$, then show that $y_n = a^n\{1+(-1)^n \sin 2ax\}^{\frac{1}{2}}$.

Sol^n- Given,

$$y = \sin ax + \cos ax$$

Differentiating n times,

$$y_n = a^n\left[\sin\left(n\frac{\pi}{2}+ax\right) + \cos\left(n\frac{\pi}{2}+ax\right)\right]$$

$$= a^n(\sin\theta + \cos\theta), \text{ where } \theta = n\frac{\pi}{2} + ax.$$

$$= a^n\{(\sin\theta+\cos\theta)^2\}^{\frac{1}{2}}$$

$$= a^n(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta)^{\frac{1}{2}}$$

$$= a^n(1+\sin 2\theta)^{\frac{1}{2}}$$

$$= a^n\{1+\sin(n\pi+2ax)\}^{\frac{1}{2}} \quad \left[\because \theta = \frac{n\pi}{2}+ax\right]$$

$$= a^n\{1+(\sin n\pi \cos 2ax + \cos n\pi \sin 2ax)\}^{\frac{1}{2}}$$

$$= a^n[1+0+(-1)^n \sin 2ax]^{\frac{1}{2}} \quad \left[\because \sin n\pi = 0,\ \cos n\pi = (-1)^n\right]$$

$$= a^n\{1+(-1)^n\sin 2ax\}^{\frac{1}{2}}$$

(Proved)

6. If $f(x) = x^n$, prove that $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \cdots + \frac{f^n(1)}{n!} = 2^n$.

Hints - $f(x) = x^n$

$\therefore f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^n$, ---, $f^n(x) = n!$

$\Rightarrow f'(1) = n$, $f''(1) = n(n-1)$, ---, $f^n(1) = n!$

Now, $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \cdots + \frac{f^n(1)}{n!}$

$= 1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \cdots + \frac{n!}{n!} = 1 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

$= (1+1)^n = 2^n$

(8) **7.** If $y = \frac{x^3}{x^2-1}$, then prove that

$$(y_n)_0 = \begin{cases} 0, & n \text{ even} \\ -n!, & n \text{ odd} \end{cases}, \quad n>1$$

Hints: $y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} = x + \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x-1}\right)$

$$\therefore y_n = 0 + (-1)^n \frac{n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

~~Let n be even i.e. n = 2K (say), K ∈ ℕ.~~

$\cancel{\therefore (y)_{2K} = (-1)^{2K}(2K)!}\ [$

$$(y_n)_0 = \frac{(-1)^n}{2} n! \left[(-1)^{n+1} + 1^{n+1}\right] = \frac{(-1)^n}{2} n! \left[(-1)^{n+1} + 1\right]$$

$n = \text{even} = 2K$ (say), $K \in \mathbb{N} \Rightarrow (y_{2K})_0 = \frac{(-1)^{2K}}{2}(2K!)\left[(-1)^{2K+1} + 1\right] = 0$

$n = \text{odd} = 2K+1$ (say), $K \in \mathbb{N} \Rightarrow (y_{2K+1})_0 = \frac{(-1)^{2K+1}}{2}(2K+1)!\left[(-1)^{2K+2} + 1\right]$

$= -(2K+1)!$

$= -(n!)$

$$\therefore (y_n)_0 = \begin{cases} 0, & \text{if } n \text{ is even} \\ -(n!), & \text{if } n \text{ is odd} \end{cases}$$

Leibnitz's (Leibniz's) Theorem

If u and v be two functions of x, each possessing derivatives upto n^{th} order, then the product $y = u \cdot v$ is derivable n times and

$$y_n = (uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \cdots$$
$$\cdots + {}^n c_r u_{n-r} v_r + \cdots + u v_n.$$

where the y_r, u_r, v_r denote the r^{th} derivatives of y, u and v respectively w.r.t. x.

Proof - Since, $y = uv$, we have

$$y_1 = u_1 v + u v_1.$$

Thus the theorem is true for $n = 1$.

Let us assume that the theorem is true for $n = m$ where $m < n$. Then

$$y_m = (uv)_m = u_m v + {}^m c_1 u_{m-1} v_1 + {}^m c_2 u_{m-2} v_2 + \cdots$$
$$\cdots + {}^m c_r u_{m-r} v_r + \cdots + u v_m$$

Differentiating both sides w.r.t x, we get

$$y_{m+1} = (u_{m+1} v + u_m v_1) + {}^m c_1 (u_m v_1 + u_{m-1} v_2)$$
$$+ {}^m c_2 (u_{m-1} v_2 + u_{m-2} v_3) + \cdots + {}^m c_r (u_{m-r+1} v_r + u_{m-r} v_{r+1})$$
$$+ \cdots + (u_1 v_m + u v_{m+1}).$$

$$= u_{m+1} v + (1 + {}^m c_1) u_m v_1 + ({}^m c_1 + {}^m c_2) u_{m-1} v_2$$
$$+ \cdots + ({}^m c_r + {}^m c_{r+1}) u_{m-r+1} v_r + \cdots + u v_{m+1}$$

$= u_{m+1}v + {}^{m+1}c_1 u_m v_1 + {}^{m+1}c_2 u_{m-1} v_2 +$
$\cdots + {}^{m+1}c_r u_{m-r+1} v_r + \cdots + u v_{m+1}$

$$\left[\because 1 + {}^m c_1 = 1+m = {}^{1+m}c_1, \quad {}^n c_r + {}^n c_{r-1} = {}^{n+1}c_r \right]$$

Thus the theorem is true for $n = m+1$, whenever it is true for $n = m$.

But it is already proved that the theorem is true for $n=1$. Hence it is true for $n = 1+1=2, 2+1=3$ and so on.

Therefore, by the principle of Mathematical induction, the theorem is true for any positive integral values of

Problems :

1. If $y = e^{-x}\cos x$, then show that $y_4 + 4y = 0$.

Sol^n - Since $y = e^{-x}\cos x$, we have

$$y_4 = (-1)^4 e^{-x}\cos x + {}^4c_1 (-1)^3 e^{-x}(-\sin x) + {}^4c_2 (-1)^2 e^{-x}(-\cos x) + {}^4c_3 (-1) e^{-x}(\sin x) + e^{-x}\cos x.$$

$$= e^{-x}\{\cos x + 4\sin x - 6\cos x - 4\sin x + \cos x)$$

$$= -4e^{-x}\cos x$$

$$= -4y$$

or, $y_4 + 4y = 0$ (Proved)

2. If $y = x^{n-1}\log x$, then show that $y_n = \frac{(n-1)!}{x}$.

Sol^n - Given

$$y = x^{n-1}\log x \quad --(1)$$

Differentiating (1) w.r.t x, we obtain

$$y_1 = (n-1)x^{n-2}\log x + x^{n-1}\cdot\frac{1}{x}$$

$$= \frac{(n-1)x^{n-1}\log x + x^{n-1}}{x}$$

$$\text{or, } xy_1 = (n-1)y + x^{n-1} \quad --(2)$$

Differentiating (2), $(n-1)$ times w.r.t x, by Leibnitz's theorem, we get

$$xy_n + {}^{n-1}C_1 \cdot 1 \cdot y_{n-1} = (n-1)y_{n-1} + (n-1)!$$

$$\text{or, } xy_n + {}^{n-1}C_1\, y_{n-1} = {}^{n-1}C_1\, y_{n-1} + (n-1)!$$

$$\text{or, } xy_n = (n-1)! \qquad [\because {}^{n-1}C_1 = n-1]$$

$$\text{or, } y_n = (n-1)!/x \quad \text{(Proved)}$$

3. Find the nth derivative of the following:

(a) $y = x^2e^{ax}$, (b) $y = e^{ax}\sin bx$, (c) $y = x^n(1-x)^n$.

Sol^n - (a) $y_n = (a^ne^{ax})x^2 + {}^nC_1(a^{n-1}e^{ax})2x + {}^nC_2(a^{n-2}e^{ax})\cdot 2$

$$= a^ne^{ax}\cdot x^2 + 2na^{n-1}e^{ax}\cdot x + \frac{n(n-1)}{2}a^{n-2}e^{ax}\cdot 2$$

$$= e^{ax}a^{n-2}\{a^2x^2 + 2nax + n(n-1)\}.$$

(b) $y_n = (a^n e^{ax}) \sin bx + {}^nc_1 (a^{n-1} e^{ax}) \cos bx \cdot b$

$+ {}^nc_2 (a^{n-2} e^{ax}) (-\sin bx) b^2 + \cdots + e^{ax} \sin(n\frac{\pi}{2} + bx) b^n$

$= e^{ax} \left[a^n \sin bx + {}^nc_1 a^{n-1} b \cdot \cos bx + {}^nc_2 a^{n-2} b^2 \sin bx \right.$

$\left. \cdots + b^n \sin(n\frac{\pi}{2} + bx) \right]$

(c) $y_n = n! (1-x)^n + {}^nc_1 \left(\frac{n!}{1!} x\right) n(1-x)^{n-1}(-1)$

$+ {}^nc_2 \left(\frac{n!}{2!} x^2\right) n(n-1)(1-x)^{n-2}(-1)^2 + \cdots$

$= n! \left[(1-x)^n - {}^nc_1 \cdot n \cdot x(1-x)^{n-1} + {}^nc_2 \frac{n(n-1)}{2!} x^2 (1-x)^{n-2} - \cdots \right]$

$= n! \left[(1-x)^n - ({}^nc_1)^2 (1-x)^{n-1} x + ({}^nc_2)^2 (1-x)^{n-2} x^2 - \cdots \right]$

4) If $y = \cos(10 \cos^{-1} x)$, show that $(1-x^2) y_{12} = 21 x y_{11}$.

Solⁿ- From $y = \cos(10 \cos^{-1} x)$, we have

$$\cos^{-1} y = 10 \cos^{-1} x$$

Differentiating w.r.t x, we get

$$-\frac{1}{\sqrt{1-y^2}} y_1 = 10 \left(-\frac{1}{\sqrt{1-x^2}}\right)$$

Hence, $\left(-\frac{y_1}{\sqrt{1-y^2}}\right)^2 = 100 \left(-\frac{1}{\sqrt{1-x^2}}\right)^2$

or, $(1-x^2) y_1^2 = 100(1-y^2)$.

Differentiating again w.r.t x, we have

$$(1-x^2) 2 y_1 y_2 - 2 x y_1^2 = 100(-2 y y_1)$$

or, $(1-x^2)y_2 - xy_1 = -100y$ $[\because 2y_1 \neq 0]$.

Differentiating above equation 10 times by Leibnitz's theorem, we get

$$\{(1-x^2)y_{12} + {}^{10}c_1\, y_{11}(-2x) + {}^{10}c_2\, y_{10}(-2)\}$$

$$-\{y_{11}\cdot x + {}^{10}c_1\, y_{10}\cdot 1\} = -100y_{10}$$

or, $(1-x^2)y_{12} - 20xy_{11} - 90y_{10} - xy_{11} - 10y_{10} + 100y_{10} = 0$

or, $(1-x^2)y_{12} = 21xy_{11}$ (Proved)

5. If $f(x) = \tan x$ and n is a positive integer, show by Leibnitz's theorem, that

$$f^{n}(0) - {}^{n}c_2\, f^{n-2}(0) + {}^{n}c_4\, f^{n-4}(0) - \cdots = \sin\left(n\frac{\pi}{2}\right)$$

Sol^n - We have,

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$\text{or, } f(x)\cos x = \sin x \quad \text{---(1)}$$

Differentiating (1) n times w.r.t x, we have

$$f^{n}(x)\cos x + {}^{n}c_1\, f^{n-1}(x)(-\sin x) + {}^{n}c_2\, f^{n-2}(x)(-\cos x)$$
$$+ {}^{n}c_3\, f^{n-3}(x)(\sin x) + {}^{n}c_4\, f^{n-4}(x)(\cos x) + \cdots$$
$$= \sin\left(n\frac{\pi}{2} + x\right)$$

Now, putting $x = 0$ on both ~~sides~~ sides, we get

$$f^{n}(0) - {}^{n}c_2\, f^{n-2}(0) + {}^{n}c_4\, f^{n-4}(0) - \cdots = \sin(n\pi/2).$$

6 If $y = e^{m\sin^{-1}x}$, show that (13)

(i) $(1-x^2)y_2 - xy_1 - m^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (m^2+n^2)y_n = 0$.

Also find $(y_n)_0$.

Soln- Given,

$$y = e^{m\sin^{-1}x} \quad \text{--- (1)}$$

Differentiating (1) w.r.t x, we get

$$y_1 = e^{m\sin^{-1}x} \cdot \frac{m}{\sqrt{1-x^2}} = \frac{my}{\sqrt{1-x^2}} \quad \text{--- (1.1)}$$

or, $y_1^2 = \dfrac{m^2y^2}{1-x^2}$

or, $(1-x^2)y_1^2 = m^2y^2 \quad \text{--- (2)}$

Differentiating (2) w.r.t x, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = m^2 \cdot 2yy_1$$

or, $(1-x^2)y_2 - xy_1 - m^2y = 0$ --- (3) (Proved)

Applying Leibnitz's theorem to differentiate (3) n times w.r.t x, we get

$$\left\{(1-x^2)y_{n+2} + {}^nc_1 y_{n+1}(-2x) + {}^nc_2 y_n(-2)\right\}$$

$$- \left\{x y_{n+1} + {}^nc_1 y_n \cdot 1\right\} - m^2 y_n = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - \dfrac{n(n-1)}{2} \cdot 2y_n - xy_{n+1} - ny_n - m^2y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (m^2+n^2)y_n = 0$ --- (4)

(Proved)

Putting $x=0$ in (1), we get

$$(y)_0 = 1.$$

Also from (1), (3) and (4)

$$(y_1)_0 = m,\ (y_2)_0 = m^2 \ \&\ (y_{n+2})_0 = (m^2+n^2)(y_n)_0 \quad \text{---}(5)$$

Now putting $n = 1, 2, 3, \ldots$ in (5), we have

$$(y_3)_0 = (m^2+1^2)(y_1)_0 = m(m^2+1^2);$$

$$(y_4)_0 = (m^2+2^2)(y_2)_0 = m^2(m^2+2^2);$$

$$(y_5)_0 = (m^2+3^2)(y_3)_0 = m(m^2+1^2)(m^2+3^2);$$

$$(y_6)_0 = (m^2+4^2)(y_4)_0 = m^2(m^2+2^2)(m^2+4^2) \text{ etc.}$$

$$\therefore (y_n)_0 = m(m^2+1^2)(m^2+3^2)\cdots\{m^2+(n-2)^2\}, \text{ when } n \text{ is odd}$$

$$= m^2(m^2+2^2)(m^2+4^2)\cdots\{m^2+(n-2)^2\}, \text{ when } n \text{ is even.}$$

7. If $x = \sin\theta$, $y = \sin k\theta$ prove that

(i) $(1-x^2)y_2 - xy_1 + k^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (k^2-n^2)y_n = 0.$

Hints.

$$x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta$$

$$y = \sin k\theta \Rightarrow \frac{dy}{d\theta} = k\cos k\theta$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{k\cos k\theta}{\cos\theta}$$

$$\Rightarrow y_1\cos\theta = k\cos k\theta \Rightarrow y_1^2\cos^2\theta = k^2\cos^2 k\theta$$

$$\Rightarrow y_1^2(1-\sin^2\theta) = k^2(1-\sin^2 k\theta) \Rightarrow y_1^2(1-x^2) = k^2(1-y^2)$$

Differentiating

$$(1-x^2)y_2 - xy_1 + k^2y = 0.$$

8. If $y=(\sin^{-1}x)^2$, then show that

(i) $(1-x^2)y_2 - xy_1 - 2 = 0$

(ii) $(1-x^2)y_{n+2} - (2n-1)xy_{n+1} - n^2y_n = 0$

Sol^n– Here,

$$y=(\sin^{-1}x)^2 \quad ---(1)$$

Differentiating (1) w.r.t x, we get

$$y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\cdot y_1 = 2\sin^{-1}x$

or, $(1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y$, by (1)

Differentiating again w.r.t x, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$$

or, $(1-x^2)y_2 - xy_1 - 2 = 0 \quad ---(2)$

(Proved)

Differentiating (2) n times by Leibnitz's theorem, we get

$$\left\{(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n\right\}$$
$$-\left\{xy_{n+1} + {}^nc_1\cdot 1\cdot y_n\right\} - 0 = 0$$

or, $(1-x^2)y_{n+2} - (2n-1)xy_{n+1} - n^2y_n = 0$ (Proved)

9. If $y=(x^2-1)^n$, then show that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

Sol^n– We have,

$$y=(x^2-1)^n \quad ---(1)$$

Differentiating (1) w.r.t 'x', we get

(ii) $$y_1 = n(x^2-1)^{n-1} \cdot 2x = \frac{2nx(x^2-1)^n}{x^2-1} = \frac{2nxy}{x^2-1}, \text{ by (1)}$$

or, $(x^2-1)y_1 = 2nxy$

Differentiating again

$$(x^2-1)y_2 + 2xy_1 = 2ny + 2nxy_1$$

$$\text{or, } (x^2-1)y_2 + 2(1-n)xy_1 - 2ny = 0 \quad \text{---(2)}$$

Differentiating (2) ---------

$$\left\{ (x^2-1)y_{n+2} + {}^nc_1(2x)y_{n+1} + {}^nc_2(2)\cdot y_n \right\}$$

$$+2(1-n)\left\{ xy_{n+1} + {}^nc_1 \cdot 1 \cdot y_n \right\} - 2ny_n = 0$$

$$\text{or, } (x^2-1)y_{n+2} + y_{n+1}\left\{2nx + 2(1-n)x\right\}$$

$$+ \frac{n(n-1)}{2}\cdot 2 \cdot y_n + ny_n - 2ny_n = 0$$

or, $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$ (Proved)

10 If $y = \sin(m\sin^{-1}x)$, then prove that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$.

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Hints - $y = \sin(m\sin^{-1}x)$

$$\Rightarrow \sin^{-1}y = m\sin^{-1}x$$

Differentiating

$$\frac{1}{\sqrt{1-y^2}}y_1 = m\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\text{or, } (1-x^2)y_1^2 - m^2(1-y^2) = 0.$$

Again differentiating

$$(1-x^2)y_2 - xy_1 + m^2 y = 0.$$

Differentiating n times

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

11. If $y = \tan^{-1}x$, then show that

(i) $(1+x^2)y_1 = 0$

(ii) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$

Also find the value of $(y_n)_0$.

Sol^n – We have

$$y = \tan^{-1}x \quad \text{---(1)}$$

Differentiating (1) w.r.t 'x' we get

$$y_1 = \frac{1}{1+x^2}$$

$$\text{or, } (1+x^2)y_1 = 1 \quad \text{---(2)}$$

(Proved)

Differentiating (2) n times by Leibnitz's theorem, we get

$$(1+x^2)y_{n+1} + {}^nC_1 \cdot 2x \cdot y_n + {}^nC_2 \cdot 2 \cdot y_{n-1} = 0$$

$$\text{or, } (1+x^2)y_{n+1} + n \cdot 2xy_n + \frac{n(n-1)}{2} \cdot 2 \cdot y_{n-1} = 0$$

$$\text{or, } (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0. \quad \text{---(3)}$$

(Proved)

Putting $x=0$ in (3), we have

$$(y_{n+1})_0 = -n(n-1)(y_{n-1})_0.$$

Replacing n by $n-1$, we get

$$(y_n)_0 = -(n-1)(n-2)(y_{n-2})_0.$$

$\therefore\ (y_{n-2})_0 = -(n-3)(n-4)(y_{n-4})_0,$

$(y_{n-4})_0 = -(n-5)(n-6)(y_{n-6})_0,$

$- - - - - - - - - - - -$

$(y_4)_0 = -3\cdot 2\cdot (y_2)_0$, when n is even

$(y_3)_0 = -2\cdot 1\cdot (y_1)_0$, when n is odd.

By successive substitution, we get

$(y_n)_0 = (-1)^{\frac{n-2}{2}}(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot (y_2)_0$, when n is even

$= (-1)^{\frac{n-2}{2}}(n-1)(n-2)(n-3)\cdots 2\cdot 1\cdot (y_1)_0$, when n is odd.

Now, $y_1 = \frac{1}{1+x^2}$ and $y_2 = -\frac{2x}{(1+x^2)^2}$ gives $(y_1)_0 = 1$ and $(y_2)_0 = 0$.

$\therefore\ (y_n)_0 = 0$, when n is even

$= (-1)^{\frac{n-1}{2}}(n-1)(n-2)(n-3)\cdots 2\cdot 1$, when n is odd.

$= (-1)^{\frac{n-1}{2}}(n-1)!$

Explanation

'n' even $\{4, 6, \ldots, (n-2), n\} \rightarrow r^{th}$ position (no. of terms is r)

$\therefore\ t_r = 4 + (r-1)\cdot 2 = n \Rightarrow r = \frac{n-2}{2}$

'n' odd $3, 5, \ldots, (n-2), n \rightarrow r^{th}$ position

$\therefore\ t_r = 3 + (r-1)\cdot 2 = n \Rightarrow r = \frac{n-1}{2}$.

Alternative process

From (1)

$$y_2 = -\frac{2x}{(1+x^2)^2}$$

$\therefore (y_1)_0 = 1, \quad (y_2)_0 = 0.$

Now putting $x=0$ in (3), we have

$$(y_{n+1})_0 = -n(n-1)(y_{n-1})_0 \quad \text{--- (4)}.$$

Putting $n = 3, 5, 7, \ldots$ successively in (4), we get

$$(y_4)_0 = -3\cdot 2\cdot (y_2)_0 = 0. \quad [\because (y_2)_0 = 0]$$

$$(y_6)_0 = -5\cdot 4\cdot (y_4)_0 = 0 \quad [\because (y_4)_0 = 0].$$

Hence, $(y_n)_0 = 0$ when n is even.

Next putting $n = 2, 4, 6, \ldots$ successively in (4), we get

$$(y_3)_0 = -2\cdot 1\cdot (y_1)_0 = (-1)\cdot 2!, \quad [\because (y_1)_0 = 1]$$

$$(y_5)_0 = -4\cdot 3\cdot (y_3)_0 = (-1)^2\cdot 4!, \quad [\because (y_3)_0 = (-1)^1 2!]$$

Similarly, $(y_7)_0 = (-1)^3\, 6!$

Hence, $(y_n)_0 = (-1)^{\frac{n-1}{2}}(n-1)!$ when n is odd.

Thus,

$$(y_n)_0 = 0 \quad \text{when } n \text{ is even}$$

$$= (-1)^{\frac{n-1}{2}}(n-1)!, \quad \text{when } n \text{ is odd},$$

12. Show that the nth derivative of $\frac{\log x}{x}$ is

$$(-1)^n \frac{n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right).$$

Sol^n- Let $y = \frac{\log x}{x} = \frac{1}{x}\cdot \log x$

$$\therefore\ y_n = \frac{(-1)^n n!}{x^{n+1}}\cdot \log x + {}^nc_1 \frac{(-1)^{n-1}(n-1)!}{x^n}\cdot\frac{1}{x}$$

$$+\ {}^nc_2 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\cdot\left(-\frac{1}{x^2}\right) + {}^nc_3\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\cdot\left(\frac{2}{x^3}\right)$$

$$+ \cdots + \frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right].$$

13 If $y = \frac{ax^2+bx+c}{1-x}$, then show that $(1-x)y_3 = 3y_2$.

Sol^n- Given,
$$y = \frac{ax^2+bx+c}{1-x}$$

or, $(1-x)y = ax^2+bx+c$.

Differentiating 3 times, we have.

$(1-x)y_3 + {}^3c_1(-1)y_2 + 0 = 0$

or, $(1-x)y_3 = 3y_2$.

14. By forming in two different ways the n^{th} derivative of x^{2n}, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \cdots = \frac{(2n)!}{(n!)^2}$$

Solⁿ - Differentiating $x^{2n} = x^n \cdot x^n$, n times,

(21)

$$\frac{(2n)!}{(2n-n)!} x^{2n-n} = n! \cdot x^n + {}^nc_1\left(\frac{n!}{1!}x\right)\cdot(nx^{n-1}) + {}^nc_2\left(\frac{n!}{2!}x^2\right)n(n-1)x^{n-2}$$
$$+ {}^nc_3\left(\frac{n!}{3!}x^3\right)n(n-1)(n-2)x^{n-3} + \cdots\cdots$$

or, $$\frac{(2n)!}{n!}x^n = x^n n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{2^2 \cdot 1^2} + \frac{n^2(n-1)^2(n-2)^2}{3^2 \cdot 2^2 \cdot 1^2} + \cdots\right]$$

or, $$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{2^2 \cdot 1^2} + \frac{n^2(n-1)^2(n-2)^2}{3^2 \cdot 2^2 \cdot 1^2} + \cdots = \frac{(2n)!}{(n!)^2}$$

15 If $y = (x + \sqrt{1+x^2})^m$, prove that

(i) $(1+x^2)y_2 + xy_1 - m^2 y = 0$

(ii) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

Also find $(y_n)_0$.

Hints: $y = (x + \sqrt{1+x^2})^m$ — (1)

$$y_1 = m\left(x + \sqrt{1+x^2}\right)^{m-1} \times \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) = \frac{m(1+\sqrt{1+x^2})^m}{\sqrt{1+x^2}}$$

or, $y_1\sqrt{1+x^2} = my$ — (2)

or, $(1+x^2)y_1^2 = m^2 y^2$

Differentiating 2 times —

$$(1+x^2)y_2 + xy_1 - m^2 y = 0 \quad \text{— (3)}$$

Differentiating n times —

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0 \quad \text{— (4)}$$

Putting $x = 0$ in (4)

$$(y_{n+2})_0 = (m^2 - n^2)(y_n)_0$$

Replacing n by $n-2$

$$(y_n)_0 = [m^2-(n-2)^2](y_{n-2})_0 \quad \text{---(5)}$$

$$\therefore (y_{n-2})_0 = [m^2-(n-4)^2](y_{n-4})_0$$

$$\cdots\cdots$$

$$(y_4)_0 = (m^2-2^2)(y_2)_0 \quad \text{, if } n \text{ is even}$$

$$(y_3)_0 = (m^2-1^2)(y_1)_0 \quad \text{, if } n \text{ is odd.}$$

By successive substitution,

$$(y_n)_0 = [m^2-(n-2)^2][m^2-(n-4)^2]\cdots(m^2-2^2)(y_2)_0 \text{, } n \text{ even}$$
$$= [m^2-(n-2)^2][m^2-(n-4)^2]\cdots(m^2-1^2)(y_1)_0 \text{, } n \text{ odd.}$$

Now $(y)_0 = 1$, $(y_1)_0 = m$, $(y_2)_0 = m^2$

$$\therefore (y_n)_0 = [m^2-(n-2)^2][m^2-(n-4)^2]\cdots(m^2-2^2)m^2 \text{, } n \text{ even}$$
$$= [m^2-(n-2)^2][m^2-(n-4)^2]\cdots(m^2-1^2)\cdot m \text{, } n \text{ odd.}$$

16) If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$, then show that

(i) $(1-x^2)y_2 - 3xy_1 - y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$.

Hints – $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$$\Rightarrow (1-x^2)y^2 = (\sin^{-1}x)^2$$

Differentiating,

$$(1-x^2)2yy_1 - 2xy^2 = 2\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$
$$\Rightarrow (1-x^2)y_1 - xy = 1$$

Differentiating

$(1-x^2)y_2 - 3xy_1 - y = 0$

Differentiating, n times.

$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$

17 If $y = a\cos(\log x) + b\sin(\log x)$, then show that

(i) $x^2y_2 + xy_1 + y = 0$

(ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

Hints → $y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$

Differentiating,

$xy_2 + y_1 = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$

$\Rightarrow x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)] = -y$

18 If $x = \cosh(\frac{1}{m}\log y)$, show that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$

Hints - $x = \cosh(\frac{1}{m}\log y) \Rightarrow y = e^{m\cosh^{-1}x}.$

$\therefore y_1 = e^{m\cosh^{-1}x} \cdot \frac{m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

or, $(x^2-1)y_1^2 = m^2y^2.$

18) Find y_n, where

(i) $y = \tan^{-1} x$, (ii) $y = \tan^{-1}\frac{1+x}{1-x}$

(iii) $y = \tan^{-1}\left(\frac{x}{a}\right)$, (iv) $y = \frac{x}{x^2+a^2}$, (v) $y = \frac{1}{1+x+x^2+x^3}$

(vi) $y = \frac{1}{(ax+b)^m}$

Solⁿ–

18 (i) $y = \tan^{-1} x$.

Differentiating w.r.t x, we have

$$y_1 = \frac{1}{1+x^2}$$

or $$y_1 = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right).$$

Differentiating $(n-1)$ times w.r.t. x,

$$y_n = \frac{1}{2i}\left[(-1)^{n-1}\frac{(n-1)!}{(x-i)^n} - (-1)^{n-1}\frac{(n-1)!}{(x+i)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[(x-i)^{-n} - (x+i)^{-n}\right].$$

Let $x = r\cos\theta$ and $1 = r\sin\theta$, where $r = \sqrt{x^2+1}$, $\cot\theta = x$

$\therefore (x+i)^{-n} = (r\cos\theta + ir\sin\theta)^{-n}$

$= r^{-n}(\cos\theta + i\sin\theta)^{-n}$

$= \frac{1}{r^n}(\cos n\theta - i\sin n\theta)$, by De Moivre's theorem.

Similarly, $(x-i)^{-n} = \frac{1}{r^n}(\cos n\theta + i\sin n\theta)$.

Now,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i \cdot r^n}\left[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)\right]$$

$$= \frac{(-1)^n (n-1)!}{2i \cdot r^n} \cdot 2i\sin n\theta$$

$$= \frac{(-1)^n (n-1)!}{\left(\frac{1}{\sin\theta}\right)^n} \sin n\theta \quad \left[\because 1 = r\sin\theta \Rightarrow r = \frac{1}{\sin\theta}\right]$$

$$= (-1)^n (n-1)! \sin^n\theta \sin n\theta.$$

18.(ii) $y = \tan^{-1}\frac{1+x}{1-x} = \tan^{-1}1 - \tan^{-1}x$

Next, similar to 18(i).

18 (iii) Hints:-

$y = \tan^{-1}(x/a)$.

$$\therefore y_1 = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{a}{x^2+a^2} = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$\therefore y_n = \frac{1}{2i}\left[(-1)^{n-1}\frac{(n-1)!}{(x-ia)^n} - (-1)^{n-1}\frac{(n-1)!}{(x+ia)^n}\right].$$

$x = r\cos\theta$, $a = r\sin\theta$,

$\therefore (x+ia)^{-n} = \frac{1}{r^n}(\cos n\theta - i\sin n\theta)$,

$(x-ia)^{-n} = \frac{1}{r^n}(\cos n\theta + i\sin n\theta)$.

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i \cdot r^n} \cdot 2i\sin n\theta$$

$$= (-1)^{n-1}\frac{(n-1)!}{a^n}\sin^n\theta\sin n\theta \quad \left[\because a = r\sin\theta \Rightarrow \frac{1}{r} = \frac{\sin\theta}{a}\right]$$

18 (iv) Hints

$$y = \frac{x}{x^2+a^2} = \frac{x}{(x+ia)(x-ia)} = \frac{1}{2}\left[\frac{1}{x+ia} + \frac{1}{x-ia}\right].$$

$$\therefore y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{(x+ia)^{n+1}} + \frac{1}{(x-ia)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[(x+ia)^{-(n+1)} + (x-ia)^{-(n+1)}\right].$$

Let $x = r\cos\theta$, $a = r\sin\theta$,

$$\therefore (x+ia)^{-(n+1)} = \frac{1}{r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

Similarly,

$$(x-ia)^{-(n+1)} = \frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right].$$

$$\therefore y_n = \frac{(-1)^n n!}{2\cdot r^{n+1}}\cdot 2\cos(n+1)\theta$$

$$= \frac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta \quad [\because a = r\sin\theta]$$

18 (v) $$y = \frac{1}{1+x+x^2+x^3} = \frac{1}{(1+x)+x(1+x^2)} = \frac{1}{(1+x)(1+x^2)}$$

$$= \frac{1}{(1+x)(x+i)(x-i)}$$

Let,

$$\frac{1}{(1+x)(x+i)(x-i)} = \frac{A}{1+x} + \frac{B}{x+i} + \frac{C}{x-i}$$

Hence

$$1 = A(x+i)(x-i) + B(1+x)(x-i) + C(1+x)(x+i).$$

Substituting $x = -1, -i, i$, we get,

$$A = \frac{1}{2},\ B = \frac{i-1}{4},\ C = -\frac{i+1}{4}.$$

$$\therefore y = \frac{1}{2} \cdot \frac{1}{x+1} + \frac{i-1}{4} \cdot \frac{1}{x+i} - \frac{i+1}{4} \frac{1}{x-i}$$

$$\therefore y_n = \frac{1}{2} (-1)^n \frac{n!}{(1+x)^{n+1}} + \frac{i-1}{4} \cdot \frac{(-1)^n n!}{(x+i)^{n+1}} - \frac{i+1}{4} \frac{(-1)^n n!}{(x-i)^{n+1}}$$

$$= \frac{1}{2} (-1)^n n! \left[\frac{1}{(1+x)^{n+1}} + \frac{i-1}{2} (x+i)^{-(n+1)} - \frac{i+1}{2} (x-i)^{-(n+1)} \right]$$

Let $x = r\cos\theta$, $1 = r\sin\theta$.

$$\therefore (x+i)^{-(n+1)} = \frac{1}{r^{n+1}} \left\{ \cos(n+1)\theta - i\sin(n+1)\theta \right\}.$$

$$(x-i)^{-(n+1)} = \frac{1}{r^{n+1}} \left\{ \cos(n+1)\theta + i\sin(n+1)\theta \right\}.$$

Remaining part — Home Work.

18 (vi) Here $y = (ax+b)^{-m}$.

Differentiating n times, we get

$$y_n = (-m)(-m-1)(-m-2)\cdots(-m-n+1)\, a^n (ax+b)^{-m-n}$$

$$= (-1)^n \frac{m(m+1)(m+2)\cdots(m+n-1)}{(ax+b)^{m+n}} \cdot a^n.$$