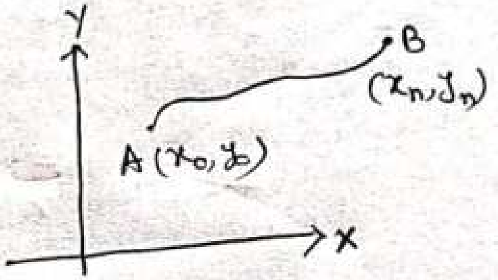
Length of a Plane Curve (Rectification)



For Cartesian equation

If the equation of the curve be $y = f(x)$, then the length of the arc AB,

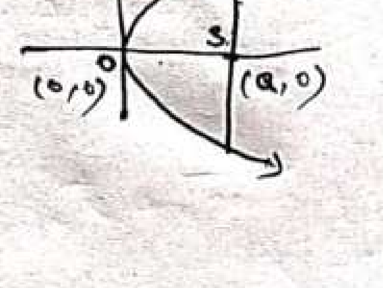
$$s = \int_{x_0}^{x_n} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_0}^{x_n} \sqrt{1+y_1^2}\, dx \quad \text{----(I)}$$

On the other hand, if equ^n be $x = g(y)$, then

$$s = \int_{y_0}^{y_n} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_0}^{y_n} \sqrt{1+x_1^2}\, dy \quad \text{---(II)}.$$

Problems –

Ex-1 Find the length of the arc of the parabola $y^2 = 4ax$ measured from the vertex to one extremity of the latus rectum.



Sol^n – Here,

$$y^2 = 4ax \quad \text{---(1)}$$

$\therefore \; 2y\frac{dy}{dx} = 4a$, or, $\frac{dy}{dx} = \frac{2a}{y}$

$$= \frac{2a}{\sqrt{4ax}}, \text{ by (1)}$$

$$= \sqrt{a/x}$$

From fig. the required length is

$$\overset{\frown}{OP} = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$x_0 = 0,\ y_0 = 0$
$x_n = a,\ y_n = 2a$

$$= \int_0^a \sqrt{1+a/x}\,dx = \int_0^a \sqrt{\frac{x+a}{x}}\,dx \quad \text{---(2)}$$

Now, $\int \sqrt{\frac{x+a}{x}}\,dx = \int \frac{x+a}{\sqrt{x(x+a)}}\,dx$

$$= \frac{1}{2}\int \frac{2x+2a}{\sqrt{x(x+a)}}\,dx = \frac{1}{2}\int \frac{2x+a+a}{\sqrt{x(x+a)}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+a}{\sqrt{x^2+ax}}\,dx + \frac{1}{2}\int \frac{a\,dx}{\sqrt{x(x+a)}}.$$

Put $x^2+ax = z^2$ in 1st integral and $x = u^2$ in 2nd

i.e, $(2x+a)dx = 2z\,dz$; $dx = 2u\,du$.

$$\therefore \int \sqrt{\frac{x+a}{x}}\,dx = \frac{1}{2}\int \frac{2z\,dz}{\sqrt{z^2}} + \frac{1}{2}\int \frac{a\cdot 2u\,du}{\sqrt{u^2(u^2+a)}}$$

$$= \frac{1}{2}\int dz + \frac{2a}{2}\int \frac{u\,du}{u\sqrt{u^2+a}}$$

$$= \frac{1}{2}z + a\int \frac{du}{\sqrt{u^2+(\sqrt{a})^2}}.$$

$$= \frac{1}{2}z + a\log\left|u + \sqrt{u^2+(\sqrt{a})^2}\right|$$

$$= \frac{1}{2}\sqrt{x^2+ax} + a\log\left|\sqrt{x}+\sqrt{x+a}\right|,$$

Now from (2)

$$\widehat{OP} = \left[\frac{1}{2}\sqrt{x^2+ax} + a\log\left|\sqrt{x}+\sqrt{x+a}\right|\right]_0^a \text{ units}$$

$$= \frac{1}{2}\sqrt{2a^2} + a\log(\sqrt{a}+\sqrt{2a}) - 0 - a\log\sqrt{a} \quad ''$$

$$= a\sqrt{2} + a\left[\log\{\sqrt{a}(1+\sqrt{2})\} - \log\sqrt{a}\right] \quad ''$$

$$= a\sqrt{2} + a\log\frac{\sqrt{a}(1+\sqrt{2})}{\sqrt{a}} = a\left[\sqrt{2}+\log(1+\sqrt{2})\right]. \quad ''$$

Ex-2 Show that the arc length of the parabola $y^2 = 16x$ measured from the vertex to an extremity of the latus rectum is $4\{\sqrt{2} + \log(1+\sqrt{2})\}$ units.

Sol^n – Same as Ex-1. [Here $a = 4$]

Ex-3. Find the lengths of the following :-

(i) the perimeter of the circle $x^2 + y^2 = a^2$

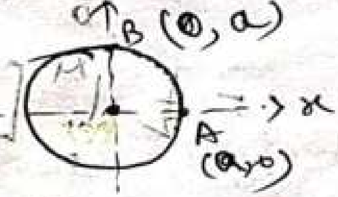
(ii) $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$; $x = 0$ & $x = 4$

(iii) the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

(iv) the arc of the semi-cubical parabola $ay^2 = x^3$ from the cusp to any point (x, y).

Sol^n (i) Here,

$$x^2 + y^2 = a^2 \quad \text{--- (1)}$$



From (1) $\quad 2x + 2y\frac{dy}{dx} = 0$ i.e. $\frac{dy}{dx} = -x/y$.

∴ The perimeter of the circle

$$= 4 \times \widehat{AB} = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$= 4\int_0^a \sqrt{1 + x^2/y^2}\, dx$$

$$= 4\int_0^a \sqrt{\frac{y^2 + x^2}{y^2}}\, dx.$$

$$= 4\int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\, dx, \quad \text{by (1)}$$

$$= 4a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a = 4a \cdot \frac{\pi}{2}$$

$$= 2a\pi \text{ units.}$$

S

(ii) Here, $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ ---(1)

$\therefore \frac{dy}{dx} = \frac{a}{2}\left(\frac{1}{a}\cdot e^{\frac{x}{a}} - \frac{1}{a}e^{-\frac{x}{a}}\right) = \frac{1}{2}\left(e^{x/a} - e^{-x/a}\right)$

$\therefore$ Required arc-length is

$$\int_0^{x_1}\sqrt{1+y_1^2}\,dx = \int_0^{x_1}\sqrt{1+\frac{1}{4}\left(e^{x/a}-e^{-x/a}\right)^2}\,dx$$

$$= \int_0^{x_1}\left[\frac{4+\left(e^{x/a}-e^{-x/a}\right)^2}{4}\right]^{1/2}dx$$

$$= \int_0^{x_1}\frac{1}{2}\left[4+\left\{e^{\frac{2x}{a}}+e^{-\frac{2x}{a}}-2e^{x/a}\cdot e^{-x/a}\right\}\right]^{1/2}dx$$

$$= \frac{1}{2}\int_0^{x_1}\left[4+e^{\frac{2x}{a}}+e^{-\frac{2x}{a}}-2\right]^{1/2}dx$$

$$= \frac{1}{2}\int_0^{x_1}\left[e^{2x/a}+e^{-\frac{2x}{a}}+2\right]^{1/2}dx$$

$$= \frac{1}{2}\int_0^{x_1}\left[e^{\frac{2x}{a}}+e^{-\frac{2x}{a}}+2\cdot e^{x/a}\cdot e^{-x/a}\right]^{1/2}dx$$

$$= \frac{1}{2}\int_0^{x_1}\left[\left(e^{x/a}+e^{-x/a}\right)^2\right]^{1/2}dx$$

$$= \frac{1}{2}\int_0^{x_1}\left(e^{x/a}+e^{-x/a}\right)dx$$

$$= \frac{1}{2}\left[\frac{e^{x/a}}{\frac{1}{a}}+\frac{e^{-x/a}}{-1/a}\right]_0^{x_1}$$

$$= \frac{a}{2}\left(e^{x_1/a}-e^{-x_1/a}\right) \text{ units}$$

Alternative Process

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) = a\cosh x/a.$$

$$\therefore \frac{dy}{dx} = a \cdot \frac{1}{a}\sinh x/a = \sinh x/a.$$

$$\therefore \text{arc-length} = \int_0^{x_1} \sqrt{1+y_1^2}\,dx = \int_0^{x_1} \sqrt{1+\sinh^2 x/a}\,dx$$

$$= \int_0^{x_1} \sqrt{\cosh^2 x/a}\,dx$$

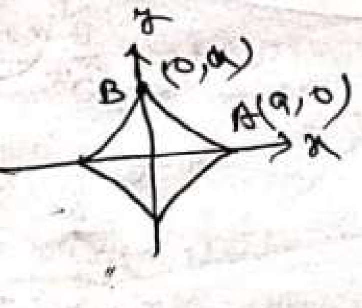
$$= \int_0^{x_1} \cosh x/a\,dx = \left[\frac{\sinh x/a}{\frac{1}{a}}\right]_0^{x_1}$$

$$= a\sinh\frac{x_1}{a}$$

$$= \frac{a}{2}\left(e^{x_1/a} - e^{-x_1/a}\right) \text{ units.}$$

3 (iii) We have,

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{---(1)}$$



From (1)

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\text{or, } \frac{dy}{dx} = -\,y^{1/3}/x^{1/3}$$

∴ The length of the perimeter of the astroid is

4 × length of the arc AB

$$= 4 \times \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^a \sqrt{1+\left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\,dx \text{ units.}$$

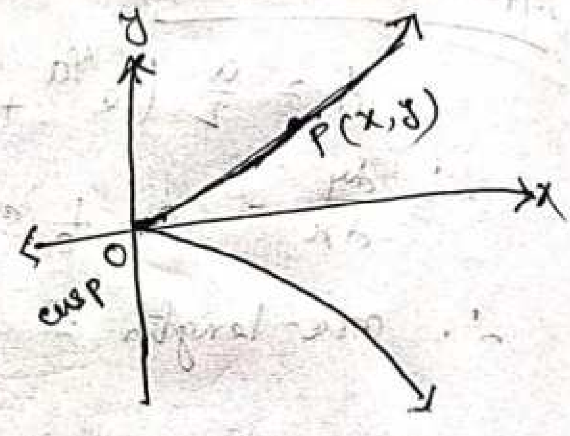
$$= 4\int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx = 4\int_0^a \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx = 4a^{1/3}\int_0^a x^{-\frac{1}{3}}\,dx = \underline{6a}$$

(by 1)

$\frac{3}{(iv)}$ Here

$$ay^2 = x^3 \quad \text{---(1)}$$



From (1)

$$2ay\frac{dy}{dx} = 3x^2$$

or $$\frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a \cdot \frac{x^{3/2}}{\sqrt{a}}} \text{, by (1)}$$

or, $$\frac{dy}{dx} = \frac{3}{2}\frac{x^{1/2}}{a^{1/2}}.$$

So the arc of the given curve from the cusp $O(0,0)$ to the point $P(x,y)$ is

$$\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{1+\frac{9}{4}\frac{x}{a}}\,dx$$

$$= \int_0^x \sqrt{\frac{4a+9x}{4a}}\,dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^x \sqrt{4a+9x}\,dx$$

$$= \frac{1}{2\sqrt{a}}\int_{4a}^{4a+9x} \frac{1}{9}\sqrt{z}\,dz \text{, putting } 4a+9x=z$$

$$\therefore dz = 9x$$

x	0	x
z	$4a$	$4a+9x$

$$= \frac{1}{8\sqrt{a}} \cdot \frac{2}{3}\left[z^{3/2}\right]_{4a}^{4a+9x}$$

$$= \frac{1}{27\sqrt{a}}\left\{(4a+9x)^{3/2} - (4a)^{3/2}\right\} \text{ units.}$$

Ex-4 Find the length of the circumference of the circle $x^2+y^2=36$.

Sol^n – Same as Ex-3 (i) (Page No – 56). Here $a=6$.

Answer – 12π units.

Ex-5 Find the length of the arc of the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ between the points $x=0$, $x=4$.

Sol^n – Same as Ex-3 (ii) (Page – 57). Here $x_1 = 4$.

Answer – $\frac{a}{2}\left(e^{4/a} - e^{-4/a}\right)$ units.

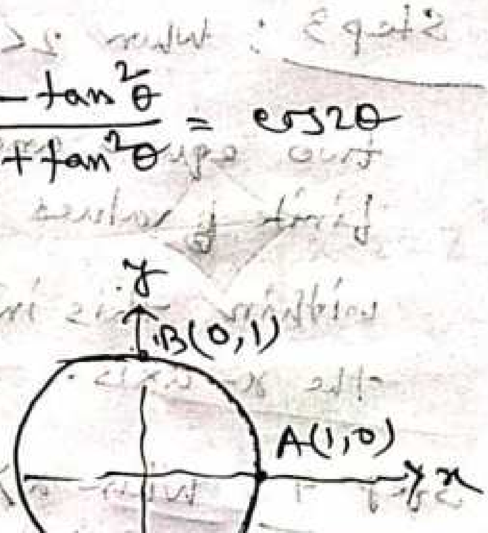
Ex-6 Show that the complete perimeter of the curve $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is 2π units.

Sol^n – Putting $t = \tan\theta$, we get $x = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

and $y = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$.

Eliminating θ, we have

$$x^2+y^2=1$$



Now same as – Ex-3 (i) (page-56)

(Here $a=1$)

Answer – 2π units.

<u>Ex-7</u> Find the whole length of the loop of the curve $9y^2 = (x-2)(x-5)^2$.

<u>Solⁿ</u>- First of all, we make a rough sketch of the curve.

From the given equation,

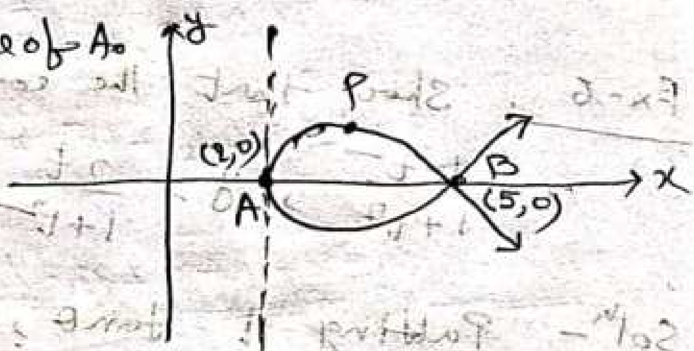
$$9y^2 = (x-2)(x-5)^2 \quad \text{---}(1)$$

we have,

$$y = \pm \frac{1}{3}\sqrt{x-2}\,(x-5) \quad \text{---}(2).$$

<u>Step 1</u>: $y = 0$, for $x = 2$ and $x = 5$.

<u>Step 2</u>: y is imaginary if $x < 2$, i.e. there is no part of the graph to the left side of A.



<u>Step 3</u>: When $2 < x < 5$, y has two equal and opposite finite values (from (2)). Thus a loop is formed within this interval, which is symmetrical about the x-axis.

<u>Step-4</u>: When $x > 5$, y has two equal and opposite values which increase in magnitude as $x \to \infty$.

Thus the shape of the curve is traced as in the adjoining figure.

From (2)

$$18y\frac{dy}{dx} = (x-5)^2 + 2(x-2)(x-5)$$
$$= (x-5)\{(x-5) + 2(x-2)\}$$
$$= 3(x-5)(x-2).$$

or, $\frac{dy}{dx} = \frac{(x-3)(x-5)}{6y}$

$= \frac{(x-3)(x-5)}{6\left\{\pm\frac{1}{3}\sqrt{x-2}\,(x-5)\right\}}$, by (2)

$= \pm\frac{x-3}{2\sqrt{x-2}}$

$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{(x-3)^2}{4(x-2)}} = \sqrt{\frac{4(x-2)+(x-3)^2}{4(x-2)}}$

$= \frac{\sqrt{4x-8+x^2-6x+9}}{2\sqrt{x-2}}$

$= \frac{\sqrt{x^2-2x+1}}{2\sqrt{x-2}} = \frac{\sqrt{(x-1)^2}}{2\sqrt{x-2}} = \frac{x-1}{2\sqrt{x-2}}.$

Now, length of the arc APB is

$\int_2^5 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_2^5 \frac{x-1}{2\sqrt{x-2}}\,dx$

$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{(z^2+2-1)\cdot 2z\,dz}{z}$ by putting $x-2=z^2$

i.e. $dx = 2z\,dz$

x	2	5
z	0	$\sqrt{3}$

$= \int_0^{\sqrt{3}} (z^2+1)\,dz$

$= \left[\frac{1}{3}z^3+z\right]_0^{\sqrt{3}}$

$= \frac{1}{3}(\sqrt{3})^3+\sqrt{3} = \frac{3\sqrt{3}}{3}+\sqrt{3} = \sqrt{3}+\sqrt{3} = 2\sqrt{3}$ units.

$\therefore$ Whole length of the loop is

$2\times$ arc length of APB

$= 4\sqrt{3}$ units

Note 1. We notice that, in the Integral $\int_2^5 \frac{x-1}{2\sqrt{x-2}}\,dx$, the integrand $\frac{x-1}{2\sqrt{x-2}}$ is undefined at $x=2$. So, one can evaluate the Integral as follows –

$$\int_2^5 \frac{x-1}{2\sqrt{x-2}}\,dx = \lim_{\varepsilon\to 0+}\int_{2+\varepsilon}^{5} \frac{x-1}{2\sqrt{x-2}}\,dx$$

$$= \lim_{\varepsilon\to 0+}\int_{\sqrt{\varepsilon}}^{\sqrt{3}} \frac{(z^2+2-1)\,2z\,dz}{2z}$$

$x-2=z^2$
$dx = 2z\,dz$

x	$2+\varepsilon$	5
z	$\sqrt{\varepsilon}$	$\sqrt{3}$

$$= \lim_{\varepsilon\to 0+}\int_{\sqrt{\varepsilon}}^{\sqrt{3}} (z^2+1)\,dz$$

$$= \lim_{\varepsilon\to 0+}\left[\frac{1}{3}z^3+z\right]_{\sqrt{\varepsilon}}^{\sqrt{3}}$$

$$= \lim_{\varepsilon\to 0+}\left[\frac{1}{3}(\sqrt{3})^3+\sqrt{3}+\frac{1}{3}(\sqrt{\varepsilon})^2+\sqrt{\varepsilon}\right]$$

$$= \frac{3\sqrt{3}}{3}+\sqrt{3}+\frac{1}{3}\cdot 0+0$$

$$= 2\sqrt{3}$$

Alternatively;

$$\int_2^5 \frac{x-1}{2\sqrt{x-2}}\,dx = \lim_{\varepsilon\to 0+}\frac{1}{2}\int_{2+\varepsilon}^{5} \frac{x-2+1}{\sqrt{x-2}}\,dx$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2}\left[\int_{2+\varepsilon}^{5} \frac{x-2}{\sqrt{x-2}}\,dx+\int_{2+\varepsilon}^{5}\frac{dx}{\sqrt{x-2}}\right]$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2}\left[\int_{2+\varepsilon}^{5}\sqrt{x-2}\,dx+\int_{2+\varepsilon}^{5}(x-2)^{-1/2}\,dx\right].$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2}\left[\frac{2}{3}(x-2)^{3/2}+2(x-2)^{1/2}\right]_{2+\varepsilon}^{5}$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{2}\left[\frac{2}{3}(3)^{3/2} + 2(3)^{1/2} - \frac{2}{3}(\varepsilon)^{3/2} - 2\varepsilon^{1/2}\right]$$ (64)

$$= \frac{1}{2}\left[\frac{2}{3}3\sqrt{3} + 2\sqrt{3} - \frac{2}{3}\cdot 0 - 2\cdot 0\right]$$

$$= 2\sqrt{3}.$$

Ex-8: Find the whole length of the loop of the curve

$$3ay^2 = x(x-a)^2$$

Sol^n - From the given equation

$$3ay^2 = x(x-a)^2 \quad \text{--- (1)}$$

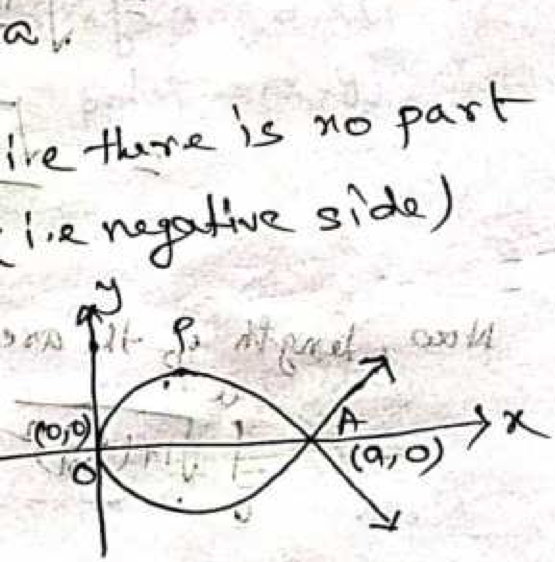
we get,

$$y = \pm\frac{1}{\sqrt{3a}}\sqrt{x}\,(x-a) \quad \text{--- (2)}.$$

First of all, we make a rough sketch of the curve.

Step-1: $y=0$, for $x=0$ and $x=a$.

Step-2: y is imaginary if $x<0$, i.e there is no part of the graph to the left side (i.e negative side) of the x-axis.



Step3: When $0<x<a$, y has two equal and opposite finite values (from (2)). Thus a loop is formed within this interval, which is symmetrical about the x-axis.

Step-4 When $x>a$, y has two equal and opposite values which increase in magnitude as $x\to\infty$.

Thus the shape of the curve is as in the adjoining figure.

From (2),

$$6ay\frac{dy}{dx} = (x-a)^2 + 2x(x-a) = (x-a)(3x-a)$$

$$\text{or, } \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

$$= \frac{(x-a)(3x-a)}{6a\left\{\pm\frac{1}{\sqrt{3a}}\sqrt{x}(x-a)\right\}}, \text{ by (2)}$$

$$= \pm\frac{\sqrt{3}}{6\sqrt{a}}\cdot\frac{3x-a}{\sqrt{x}}$$

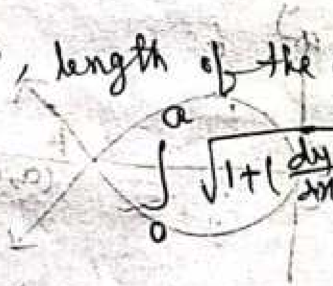
$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{3}{36a}\cdot\frac{(3x-a)^2}{x}}$$

$$= \sqrt{1+\frac{(3x-a)^2}{12ax}}$$

$$= \sqrt{\frac{12ax+9x^2-6ax+a^2}{12ax}}$$

$$= \frac{\sqrt{9x^2+6ax+a^2}}{\sqrt{12ax}} = \frac{\sqrt{(3x+a)^2}}{2\sqrt{3ax}} = \frac{3x+a}{2\sqrt{3ax}}$$

Now, length of the arc OPA is



$$\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^a \frac{3x+a}{2\sqrt{3ax}}\,dx.$$

Since, $$\int \frac{3x+a}{2\sqrt{3ax}}\,dx = \int \frac{3x}{2\sqrt{3ax}}\,dx + \int \frac{a}{2\sqrt{3ax}}\,dx$$

$$= \frac{\sqrt{3}}{2}\int \sqrt{x}\,dx + \frac{\sqrt{a}}{2\sqrt{3}}\int x^{-1/2}\,dx$$

$$= \frac{\sqrt{3}}{2\sqrt{a}}\cdot\frac{2}{3}x^{3/2} + \frac{\sqrt{a}}{2\sqrt{3}}\cdot 2x^{1/2}$$

$$= \frac{1}{\sqrt{3a}}x^{3/2} + \frac{\sqrt{a}}{\sqrt{3}}x^{1/2}$$

it follows that

$$\int_0^a \frac{3x+a}{2\sqrt{3ax}}\,dx = \left[\frac{1}{\sqrt{3a}}x^{3/2} + \sqrt{\frac{a}{3}}\,x^{1/2}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\cdot a^{3/2} + \sqrt{\frac{a}{3}}\,a^{1/2}$$

$$= \frac{1}{\sqrt{3a}}\cdot a\sqrt{a} + \sqrt{\frac{a}{3}}\cdot\sqrt{a}$$

$$= \frac{a}{\sqrt{3}} + \frac{a}{\sqrt{3}} = \frac{2a}{\sqrt{3}} = \frac{2\sqrt{3}}{3}a.$$

$\therefore$ The whole length of the loop is

$= 2\times$ length of the arc OPA

$= \frac{4\sqrt{3}}{3}a$ units.

Ex-9 If s be the length of an arc of $3ay^2 = x(x-a)^2$ measured from the origin to the point (x, y), show that $3s^2 = 4x^2 + 3y^2$.

Sol^n- Proceed same as Ex-8,

$\therefore$ The length of the arc of the given curve measured from the origin to the point (x,y) is given by

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \cdots\cdots = \left[\frac{1}{\sqrt{3a}}x^{3/2} + \sqrt{\frac{a}{3}}\,x^{1/2}\right]_0^x$$

$$= \frac{1}{\sqrt{3a}}x^{3/2} + \sqrt{a/3}\,x^{1/2}$$

$$= \frac{\sqrt{x}}{\sqrt{3a}}(x+a)$$

or, $s^2 = \frac{x}{3a}(x+a)^2$

or, $3as^2 = x(x+a)^2$

$= x\{(x-a)^2 + 4ax\}$

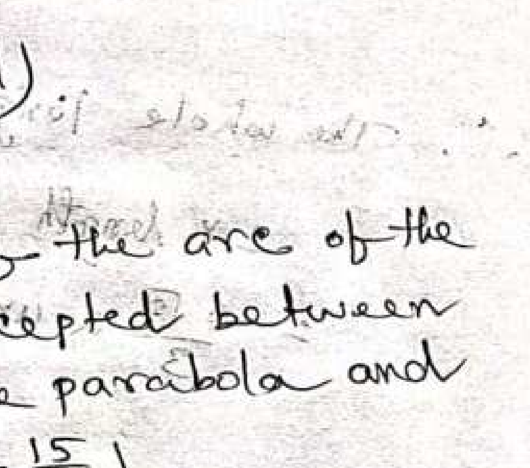
$= x(x-a)^2 + 4ax^2$

$= 3ay^2 + 4ax^2$, $\left[\because 3ay^2 = x(x-a)^2\right]$

or, $3as^2 = a(3y^2 + 4x^2)$

or, $3s^2 = 3y^2 + 4x^2$ (Proved)

Ex-10 Show that the lengths of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the st. line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$.



Soln - The given equations are

$y^2 = 4ax$ --- (1)

& $3y = 8x$ --- (2)

Solving (1) & (2) we get the points of intersection as $O(0,0)$, $A\left(\frac{9a}{16}, \frac{3a}{2}\right)$

from (1)

$$2y\frac{dy}{dx} = 4a \quad \text{or,} \quad \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2\sqrt{ax}} = \sqrt{\frac{a}{x}}.$$

~~Now, the length of the arc of the parabola (b)~~

Now, required length is

$$\int_0^{9a/16} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(63)

$$= \int_0^{\frac{9a}{16}} \sqrt{\frac{a+x}{x}}\, dx$$

$$= \left[\sqrt{x^2+ax} + a \log\left| \sqrt{x} + \sqrt{x+a} \right| \right]_0^{\frac{9a}{16}} \quad , \left[\begin{array}{l}\text{See Ex-1} \\ \text{page-55}\end{array}\right]$$

$= \text{------}$

$$= a\left(\log 2 + \frac{15}{16}\right) \text{ units}$$

<u>Ex-11</u> Show that the length of the arc of the curve $y = \log \sec x$, between $x=0$ and $x = \pi/3$ is $\log(2+\sqrt{3})$ units.

<u>Hints</u> – $y = \log \sec x \Rightarrow \frac{dy}{dx} = \frac{\sec x \cdot \tan x}{\sec x} = \tan x$.

$$\therefore s = \int_0^{\frac{\pi}{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx$$

$$= \left[\log\left|\sec x + \tan x\right|\right]_0^{\pi/3}$$

$$= \log\left(2+\sqrt{3}\right) \text{ units}$$

Length of a curve in parametric form

If the equation of a curve be $x = \psi(t)$, $y = \phi(t)$, t being parameter, length of the arc between t_1, t_2-points is,

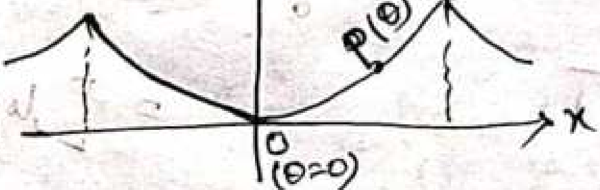
$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Problems

Ex-12 Determine the length of an arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, measured from the vertex (i.e. the origin).

Sol^n – Here,

$$x = a(\theta + \sin\theta)$$
$$y = a(1 - \cos\theta)$$



$\therefore \dfrac{dx}{d\theta} = a(1 + \cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$ and so

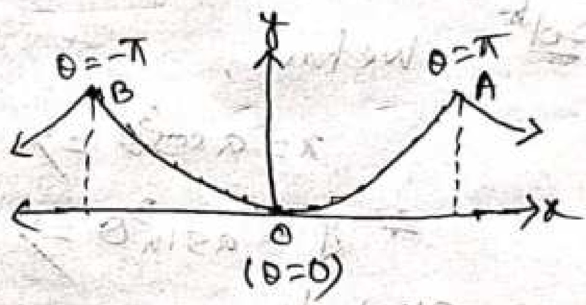
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$
$$= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$$
$$= a^2(2 + 2\cos\theta)$$
$$= 2a^2(1 + \cos\theta)$$
$$= 2a^2 \cdot 2\cos^2\theta/2 = 4a^2\cos^2\theta/2$$

So, the required length from $\theta = 0$ (origin) to any point θ is

$$\int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^\theta 2a\cos\theta/2\, d\theta = 4a\sin\theta/2.$$

Ex-13. Determine the length of one arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Sol^n – let AOB be an arc of the cycloid



$x = a(\theta + \sin\theta)$,

$y = a(1 - \cos\theta)$.

At A, $\theta = \pi$; at O, $\theta = 0$ and at B, $\theta = -\pi$.

Here, $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$.

Now, $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2\cos^2\theta/2$ (see previous example)

Hence, length of the arc AOB is

$$\int_{-\pi}^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_{-\pi}^{\pi} 2a\cos\theta/2 \, d\theta$$

$$= 2\int_0^{\pi} 2a\cos\theta/2 \, d\theta$$ [$\because$ $2a\cos\theta/2$ is an even function]

$$= 4a\left[2\sin\theta/2\right]_0^{\pi}$$

$= 8a$ units.

Ex-14 Find the length of arc of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ between $\theta = 0$ and $\theta = \pi/2$.

Sol^n – Here, $\frac{dx}{d\theta} = e^{\theta}(\sin\theta + \cos\theta)$, $\frac{dy}{d\theta} = e^{\theta}(\cos\theta - \sin\theta)$.

$\therefore \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \text{-----} = \sqrt{2e^{2\theta}} = \sqrt{2}\, e^{\theta}$

So required length $= \int_0^{\pi/2} \sqrt{2}e^{\theta}\, d\theta = \sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$ units.

Ex-15 Show that the total length of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ is $6a$ units.

Solⁿ - We have,

$$x = a\cos^3\theta \Rightarrow \cos\theta = \frac{x^{1/3}}{a^{1/3}}$$

$$\& \ y = a\sin^3\theta \Rightarrow \sin\theta = \frac{y^{1/3}}{a^{1/3}}.$$

Eliminating θ,

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{a^{2/3}} = 1$$

$$\Rightarrow x^{2/3} + y^{2/3} = a^{2/3}$$

Next See Ex - 3 (iii), page No - 58.

Ex-16 Prove that the length of the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$ is $4\sqrt{3}$ units.

Solⁿ - Here

$$x = t^2 \quad \text{---}(1)$$

$$\& \ y = t - \frac{t^3}{3} \quad \text{---}(2)$$

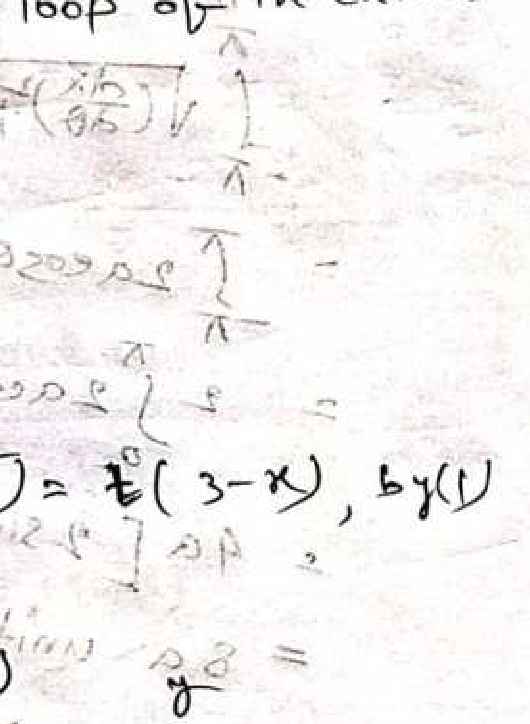
From (2),

$$3y = 3t - t^3 = t(3 - t^2) = t(3 - x), \text{ by (1)}$$

or, $9y^2 = t^2(3-x)^2$

or, $9y^2 = x(3-x)^2$, by (1) ---(3)

~~Next sketch~~ Next sketch the curve by following Ex-7, 8.



From (3)

$$\frac{dy}{dx} = \frac{(3-x)(1-x)}{6y}$$

or, $\left(\frac{dy}{dx}\right)^2 = \frac{(3-x)^2(1-x)^2}{36y^2}$

$$= \frac{(3-x)^2(1-x)^2}{36 \cdot \frac{x(3-x)^2}{9}} = \frac{\cancel{(3-x)^2}(1-x)^2}{4x}.$$

∴ The length of the loop = 2 × length of the arc OPA.

$$= 2\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^3 \sqrt{1+\frac{\cancel{(3-x)}(1-x)^2}{4x}}\,dx$$

$$= 2\int_0^3 \sqrt{\frac{4x+\cancel{(3-x)}(1-x)^2}{4x}}\,dx$$

$$= 2\int_0^3 \frac{1}{2}\sqrt{\frac{(1+x)^2}{x}}\,dx$$

$$\int_0^3 \frac{1+x}{\sqrt{x}}\,dx = \int_0^3\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)dx = 4\sqrt{3} \text{ units.}$$

(Proved)

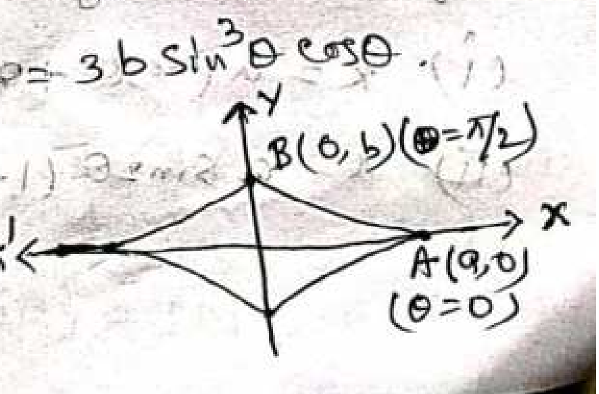
Ex-17 Find the perimeter of the hypo-cycloid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

Sol^n– consider the parametric equation of the curve

$$x = a\cos^3\theta,\; y = b\sin^3\theta.$$

∴ $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\; \frac{dy}{d\theta} = 3b\sin^3\theta\cos\theta.$



At A, $a\cos^3\theta = a \Rightarrow \theta = 0$

and at B, $a\cos^3\theta = 0 \Rightarrow \theta = \pi/2$

Now, $\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2\cos^4\theta\sin^2\theta+9b^2\sin^4\theta\cos^2\theta}$

$$= 3\sin\theta\cos\theta\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}.$$

∴ The length of the curve is

4 × length of the arc AB.

$$= 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 4\int_0^{\pi/2} 3\sin\theta\cos\theta\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$

$$= 12\int_a^b z\cdot\frac{z\,dz}{b^2-a^2}$$

Put $a^2\cos^2\theta+b^2\sin^2\theta=z^2$

$\therefore (-2a^2\cos\theta\sin\theta+2b^2\sin\theta\cos\theta) = 2z\,dz$

$\Rightarrow \sin\theta\cos\theta = \frac{z\,dz}{b^2-a^2}$

θ	0	$\pi/2$
z	a	b

$$= \frac{12}{b^2-a^2}\int_a^b z^2\,dz$$

$$= \frac{12}{b^2-a^2}\left[\frac{z^3}{3}\right]_a^b$$

$$= 4\cdot\frac{b^3-a^3}{b^2-a^2} = 4\cdot\frac{(b-a)(b^2+ab+a^2)}{(b-a)(b+a)} = 4\cdot\frac{a^2+ab+b^2}{a+b}$$ units.

Ex-18 Find the length of the arc of the following curves from $\theta=0$ to any point —

(i) $x = a(\cos\theta+\theta\sin\theta)$, $y = a(\sin\theta-\theta\cos\theta)$

(ii) $x = a\sin 2\theta(1+\cos 2\theta)$, $y = a\cos 2\theta(1-\cos 2\theta)$.

Solⁿ – (i) Here,

$$x = a(\cos\theta + \theta\sin\theta) \; \& \; y = a(\sin\theta - \theta\cos\theta)$$

$$\therefore \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta.$$

$$\& \; \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta.$$

$$\therefore \text{Length} = \int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2\theta^2}\, d\theta = \int_0^\theta a\theta\, d\theta = \frac{1}{2}a\theta^2.$$

(ii) Here,

$$x = a\sin 2\theta(1 + \cos 2\theta)$$

$$= a(\sin 2\theta + \sin 2\theta\cos 2\theta)$$

$$= \frac{a}{2}(2\sin 2\theta + 2\sin 2\theta\cos 2\theta)$$

$$= \frac{a}{2}(2\sin 2\theta + \sin 4\theta)$$

$$\therefore \frac{dx}{d\theta} = \frac{a}{2}(4\cos 2\theta + 4\cos 4\theta) = 2a(\cos 2\theta + \cos 4\theta).$$

Again, $y = a\cos 2\theta(1 - \cos 2\theta)$

$$= \frac{a}{2}(2\cos 2\theta - 2\cos^2 2\theta)$$

$$= \frac{a}{2}[2\cos 2\theta - (1 + \cos 4\theta)] = \frac{a}{2}(2\cos 2\theta - \cos 4\theta - 1)$$

$$\therefore \frac{dy}{d\theta} = \frac{a}{2}(-4\sin 2\theta + 4\sin 4\theta) = 2a(\sin 4\theta - \sin 2\theta)$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2\left\{(\cos 2\theta + \cos 4\theta)^2 + (\sin 4\theta - \sin 2\theta)^2\right\}$$

$$= 4a^2\left\{\cos^2 2\theta + \cos^2 4\theta + 2\cos 2\theta\cos 4\theta + \sin^2 4\theta + \sin^2 2\theta - 2\sin 4\theta\sin 2\theta\right\}$$

$$= 4a^2\left\{1 + 1 + 2(\cos 4\theta\cos 2\theta - \sin 4\theta\sin 2\theta)\right\}$$

$$= 4a^2\left\{2 + 2\cos(4\theta + 2\theta)\right\}$$

$$= 8a^2(1 + \cos 6\theta)$$

$$= 8a^2 \cdot 2\cos^2 3\theta = 16a^2\cos^2 3\theta.$$

$$\therefore \text{Length} = \int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{16a^2\cos^2 3\theta}\, d\theta = 4a\int_0^\theta \cos 3\theta\, d\theta = \frac{4}{3}a\sin 3\theta.$$

Length of a curve in polar system

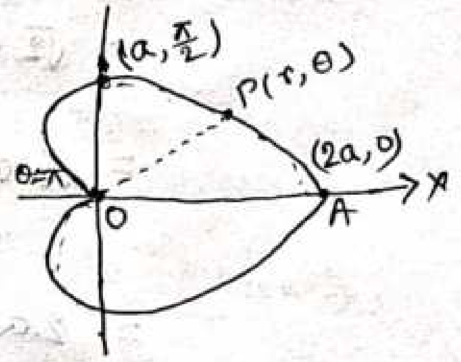
If the equation of a curve in polar coordinate system be $r = f(\theta)$, length of arc between the points whose coordinates are (r_1, θ_1) & (r_2, θ_2) is given by

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta,$$

or $$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

Ex-19 Show that the total length of the cardioide $r = a(1+\cos\theta)$ is $8a$ units.

Sol^n- Here,



$$r = a(1+\cos\theta).$$

$$\therefore \frac{dr}{d\theta} = -a\sin\theta. \quad \text{---(1)}$$

As we know, at A, $\theta = 0$ and at O, $\theta = \pi$, so the length of APO is

$$\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

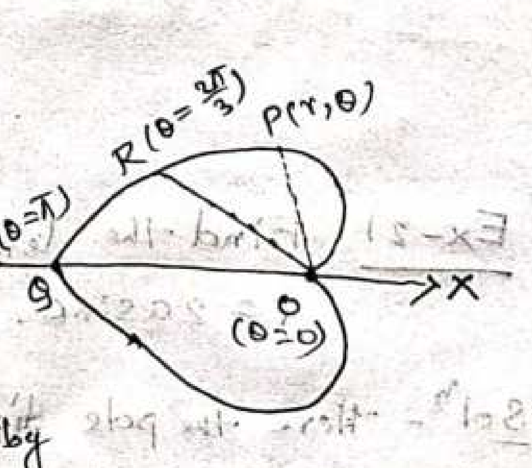
$$= \int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta \quad \text{.., by (1)}$$

$$= 2a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 2a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 4a \text{ units.}$$

$\therefore$ The total length of the cardioide is $2 \times 4a = 8a$ units.

Ex-20 Find the perimeter of the cardioide $r = a(1-\cos\theta)$. Show that the arc of the upper half is bisected at $\theta = \frac{2\pi}{3}$.

Sol^n. Here



$$r = a(1-\cos\theta).$$

$$\therefore \frac{dr}{d\theta} = a\sin\theta.$$

Hence, the length of any arc OP, measured from the origin is given by

$$\int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= a\int_0^\theta \sqrt{2-2\cos\theta}\, d\theta$$

$$= \sqrt{2}\, a\int_0^\theta \sqrt{1-\cos\theta}\, d\theta$$

$$= \sqrt{2}\, a\int_0^\theta \sqrt{2\sin^2\theta/2}\, d\theta$$

$$= 2a\int_0^\theta \sin\theta/2\, d\theta = 2a\left[-2\cos\theta/2\right]_0^\theta$$

$$= 4a(1-\cos\theta/2).$$

By putting $\theta=\pi$, we get the length of the uppere half of the curve as, $4a(1-\cos\frac{\pi}{2}) = 4a$ units.

$\therefore$ The length of the entire perimeter of the curve is $2\times 4a$ units $= 8a$ units.

Now, the arc length of OR where R $(\theta=\frac{2\pi}{3})$ be a point on the upper part, is given by

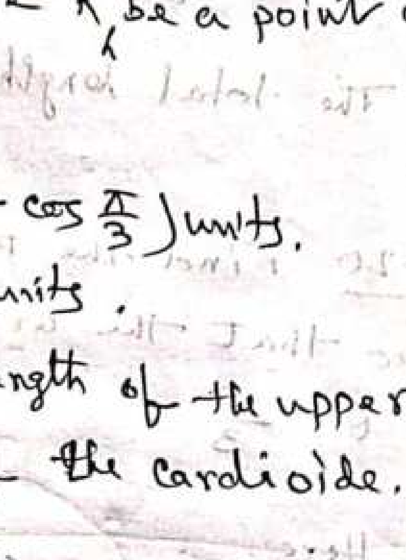
$$4a\left(1-\cos\frac{\frac{2\pi}{3}}{2}\right)\text{ units} = 4a\left(1-\cos\frac{\pi}{3}\right)\text{units}.$$

$= 2a$ units.

= half length of the upper part of the cardioide.

Ex-21 Find the length of a quadrant of the circle $r = 2a\sin\theta$.

Sol^n - Here the pole lies on the circumference of the circle and polar axis is tangent to it, as shown in the figure.



$\therefore A \equiv (a, \pi/2), B \equiv (a\sqrt{2}, \frac{\pi}{4})$,

Hence, the length of a quadrant OB is

$$\int_0^{\frac{\pi}{4}} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/4} \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta}\, d\theta$$

$$= 2a\int_0^{\pi/4} d\theta = 2a \cdot \frac{\pi}{4} = \frac{\pi a}{2} \text{ units.}$$

Ex-22 Find the length of the arc of the parabola $r(1+\cos\theta)=2$ from $\theta = 0$ to $\pi/2$.

Soln – We have,

$$r(1+\cos\theta) = 2$$

$$\text{or, } r \cdot 2\cos^2\theta/2 = 2$$

$$\text{or } r = \sec^2\theta/2 \quad \text{--- (1)}$$

From (1)

$$\frac{dr}{d\theta} = 2\cdot\sec\theta/2 \cdot \sec\theta/2 \cdot \tan\theta/2 \cdot \frac{1}{2}$$

$$= \tan\theta/2 \cdot \sec^2\theta/2 .$$

$\therefore$ Required length of the arc of the parabola.

$$= \int_0^{\pi/2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\sec^4\theta/2 + \tan^2\theta/2 \cdot \sec^4\theta/2}\, d\theta$$

$$= \int_0^{\pi/2} \sec^2\frac{\theta}{2} \cdot \sqrt{1+\tan^2\theta/2}$$

$$= \frac{1}{2}\int_0^1 \sqrt{1+z^2}\, dz$$

Put $\tan\theta/2 = z$

$\therefore \sec^2\theta/2\, d\theta = dz/2$

θ	0	$\pi/2$
z	0	1

$$= \frac{1}{2}\left[\frac{z}{2}\sqrt{1+z^2} + \frac{1}{2}\log|z+\sqrt{z^2+1}|\right]_0^1$$

$$= \sqrt{2} + \log(1+\sqrt{2}) \text{ . units.}$$

Ex-23 Find the length of the arc of the equi-angular spiral $r = ae^{\theta\cot\alpha}$ between the radii vectors r_1 & r_2.

Solⁿ – We have.

$$r = ae^{\theta\cot\alpha}$$

$$\text{or } \frac{r}{a} = e^{\theta\cot\alpha}$$

$$\text{or, } \log\frac{r}{a} = \theta\cot\alpha$$

Hence, $\frac{a}{r}\cdot\frac{1}{a} = \cot\alpha \cdot \frac{d\theta}{dr}$

$$\text{or, } r\frac{d\theta}{dr} = \tan\alpha.$$

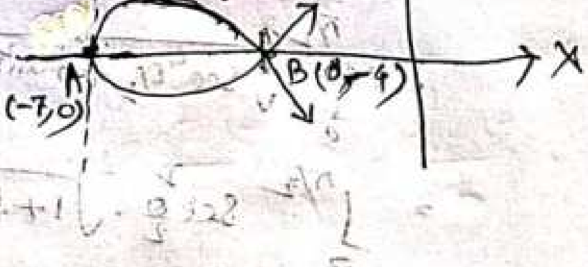
∴ Required length

$$\int_{r_1}^{r_2} \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)}\, dr$$

$$= \int_{r_1}^{r_2} \sqrt{1+\tan^2\alpha}\, dr = \sec\alpha \int_{r_1}^{r_2} dr = \sec\alpha\,(r_2 - r_1).$$

Ex-24 Find the length of the loop of the curve
$$9y^2 = (x+7)(x+4)^2.$$

Hints – To sketch the curve see Ex-7, 8 (Page-61, 64)



$$9y^2 = (x+7)(x+4)^2 \quad \text{---(1)}$$

$$\therefore 18y\frac{dy}{dx} = 3(x+4)(x+6)$$

$$\text{or, } \frac{dy}{dx} = \frac{1}{6}\,\frac{(x+4)(x+6)}{y}$$

$$\text{or, } \left(\frac{dy}{dx}\right)^2 = \frac{1}{36}\,\frac{(x+4)^2(x+6)^2}{y^2} = \frac{1}{36}\cdot\frac{(x+4)^2\cdot(x+6)^2}{\frac{1}{9}\cdot(x+7)(x+4)^2}$$

$$= \frac{1}{4}\,\frac{(x+6)^2}{x+7}.$$

$\therefore$ Required length

$= 2 \times \widehat{APB}$

$= 2 \int_{-7}^{-4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= 2 \int_{-7}^{-4} \sqrt{1+\frac{1}{4}\frac{(x+6)^2}{(x+7)}}\, dx$

$= 2 \int_{-7}^{-4} \sqrt{\frac{(x+8)^2}{4(x+7)}}\, dx$

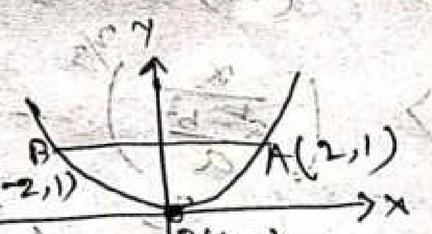
$= 2 \int_{-7}^{-4} \frac{1}{2} \cdot \frac{x+8}{\sqrt{x+7}}\, dx$

$= \int_{-7}^{-4} \frac{x+7+1}{\sqrt{x+7}}\, dx = \int_{-7}^{-4} \left[\sqrt{x+7} + \frac{1}{\sqrt{x+7}}\right] dx$

$= \left[\frac{2}{3}(x+7)^{3/2} + 2(x+7)^{1/2}\right]_{-7}^{-4}$

$= 4\sqrt{3}$ units.

Ex-25 Find the arc of the parabola $x^2 = 4y$ between the points where $y = 1$



Hints - $x^2 = 4y$ ---(1)

$\therefore 2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = x/2$

For, $y = 1$, from (1) $x^2 = 4 \Rightarrow x = \pm 2$.

So here we have to find out the length of the arc AOB where $A \equiv (2,1)$ & $B \equiv (-2,1)$.

$\therefore$ Required length $= 2 \times$ length of the arc OA.

$$= 2\times\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^2 \sqrt{1+x^2/4}\,dx = \int_0^2 \sqrt{x^2+2^2}\,dx$$

$$= \left[\frac{x\sqrt{x^2+2^2}}{2} + \frac{2^2}{2}\log\{x+\sqrt{x^2+2^2}\right]_0^2$$

$$= \frac{2}{2}\sqrt{2^2+2^2} + 2\left[\log(2+\sqrt{2^2+2^2}) - \log 2\right]$$

$$= \sqrt{2\cdot 2^2} + 2\left[\log(2+\sqrt{2\cdot 2^2}) - \log 2\right]$$

$$= 2\sqrt{2} + 2\left[\log(2+2\sqrt{2}) - \log 2\right]$$

$$= 2\left[\sqrt{2} + \log\left(\frac{2+2\sqrt{2}}{2}\right)\right]$$

$$= 2\left[\sqrt{2} + \log(1+\sqrt{2})\right] \text{ units.}$$

<u>Ex-26</u> Find the length of the perimeter of the curve

$$(ax)^{2/3} + (yb)^{2/3} = (a^2-b^2)^{2/3}$$

<u>Hints</u>- $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$

$$\Rightarrow \left(\frac{x}{\frac{a^2-b^2}{a}}\right)^{2/3} + \left(\frac{y}{\frac{a^2-b^2}{b}}\right)^{2/3} = 1$$

$$\Rightarrow \left(\frac{x}{\alpha}\right)^{2/3} + \left(\frac{y}{\beta}\right)^{2/3} = 1, \quad \alpha = \frac{a^2-b^2}{a}, \; \beta = \frac{a^2-b^2}{b}.$$

Proceed as Ex-17 (Page No-73)

$$\text{length} = 4\cdot\frac{\alpha^2+\alpha\beta+\beta^2}{\alpha+\beta}$$

Now, $\alpha+\beta = \frac{a^2-b^2}{a} + \frac{a^2-b^2}{b} = (a^2-b^2)\frac{(a+b)}{ab}$.

& $\alpha^2+\alpha\beta+\beta^2 = \left(\frac{a^2-b^2}{a}\right)^2 + \left(\frac{a^2-b^2}{a}\right)\left(\frac{a^2-b^2}{b}\right) + \left(\frac{a^2-b^2}{b}\right)^2$

$= (a^2-b^2)^2\left(\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}\right)$

$= (a^2-b^2)^2\frac{(b^2+ab+a^2)}{a^2b^2}$

$\therefore$ Length $= 4\cdot\frac{\alpha^2+\alpha\beta+b^2}{\alpha+\beta}$

$= 4\cdot(a^2-b^2)^2\frac{(b^2+ab+a^2)}{a^2b^2} \times \frac{ab}{(a^2-b^2)(a+b)}$

$= 4\left(\frac{a^2-b^2}{a+b}\cdot\frac{a^2+ab+b^2}{ab}\right)$

$= 4\cdot\frac{(a-b)(a^2+ab+b^2)}{ab}$

$= 4\frac{a^3-b^3}{ab} = 4\left(\frac{a^2}{b}-\frac{b^2}{a}\right)$ units.

Ex-27 If for a curve $x\sin\theta+y\cos\theta = f'(\theta)$ and $x\cos\theta - y\sin\theta = f''(\theta)$, prove that

$s = f(\theta)+f''(\theta)+K$, K being a constant.

Sol^n- We have

$x\sin\theta + y\cos\theta = f'(\theta)$ --- (1)

$x\cos\theta - y\sin\theta = f''(\theta)$ --- (2)

Solving (1) and (2), we get

$$x = \sin\theta \cdot f'(\theta) + \cos\theta \cdot f''(\theta)$$

& $y = \cos\theta \cdot f'(\theta) - \sin\theta\, f''(\theta)$ (H.W)

Now, $\dfrac{dx}{d\theta} = \cos\theta\, f'(\theta) + \sin\theta \cdot f''(\theta) - \sin\theta\, f''(\theta) + \cos\theta \cdot f'''(\theta)$

$$= \cos\theta \{ f'(\theta) + f'''(\theta) \}$$

& $\dfrac{dy}{d\theta} = -\sin\theta \{ f'(\theta) + f'''(\theta) \}$.

$\therefore \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2} = \sqrt{\{f'(\theta) + f'''(\theta)\}^2 (\cos^2\theta + \sin^2\theta)}$

$$= f'(\theta) + f'''(\theta).$$

So, length $s = \displaystyle\int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^\theta \{ f'(\theta) + f'''(\theta) \} d\theta$

$$= f(\theta) + f''(\theta) + K, \quad [K = \text{constant}].$$

Ex-28 Prove that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$ units.

Sol^n – We have

$$x^2(a^2 - x^2) = 8a^2y^2$$

or, $y^2 = \dfrac{1}{8a^2} x^2 (a^2 - x^2)$ ---- (1)

or, $y = \pm \dfrac{1}{2\sqrt{2} \cdot a} x \sqrt{a^2 - x^2}$

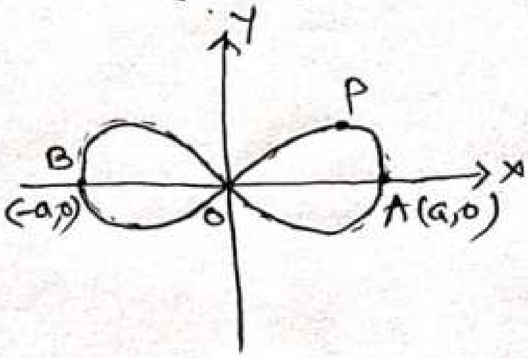
$= \pm \dfrac{1}{2\sqrt{2} \cdot a} x \sqrt{(a-x)(a+x)}$ ---- (2)

First of all, we make a rough sketch of the curve :

Step-1 $y = 0$, for $x = \pm a, 0$.

step-2 y is imaginary if $x > a$ or $x < a$; i.e there are no parts of the curve to the right side of $A(a,0)$ as well as to the left side of $B(0,-a)$.

step-3 When $-a < x < a$, y has two equal and opposite finite values (from (2)). Thus, two loops are formed between $x = -a$ and $x = a$, which are symmetrical about the x-axis.



So, the shape of the curve is traced as in the adjoing figure.

From (1)

$$\frac{dy}{dx} = \pm\frac{1}{2\sqrt{2}a}\left[\sqrt{a^2-x^2} + x\cdot\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\right] = \pm\frac{1}{2\sqrt{2}a}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

& so

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left\{1 + \frac{1}{8a^2}\cdot\frac{(a^2-2x^2)^2}{(a^2-x^2)}\right\}^{1/2}$$

$$= \frac{1}{2\sqrt{2}a}\left(-2\sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}\right) \quad \text{(H.W)}.$$

$\therefore$ Whole length of the curve

$= 4 \times$ arc length of OPA

$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\cdot\frac{1}{2\sqrt{2}a}\left\{-2\int_0^a \sqrt{a^2-x^2}\,dx + a^2\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\right\}.$$

$$= \frac{\sqrt{2}}{a}\left\{-2\left[\frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a + a^2\left[\sin^{-1}\frac{x}{a}\right]_0^a\right\}$$

$$= \frac{\sqrt{2}}{a}\left\{-2\left[0 - \frac{a^2}{2}\sin^{-1}1 - 0\right] + a^2\cdot\sin^{-1}1\right\}$$

$$= \frac{\sqrt{2}}{a}\cdot\left(a^2\cdot\pi/2 + a^2\,\pi/2\right) = \pi a\sqrt{2} \text{ units.}$$