Theorem: Every nth degree algebraic equation has exactly n roots.

Proof: Let us consider following nth degree algebraic equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots + a_{n-1}x + a_n = 0 \longrightarrow (1)$$

where $a_0 \neq 0$

Since $f(x) = 0$ is an algebraic equation, therefore by fundamental theorem of algebra it must have a root, say α_1

Hence $(x-\alpha_1)$ is a factor of $f(x)$

$$\therefore \quad f(x) = (x-\alpha_1)\,\phi_1(x) \longrightarrow (2)$$

where $\phi_1(x)$ is an algebraic equation of degree $(n-1)$

Hence by fundamental theorem of algebra it must have a root, α_2, say

Therefore $(x-\alpha_2)$ must be a factor of $\phi_1(x)$

$$\therefore \quad \phi_1(x) = (x-\alpha_2)\,\phi_2(x) \longrightarrow (3)$$

where $\phi_2(x)$ is a polynomial of degree $(n-2)$.

From (2) and (3) we get

$$f(x) = (x-\alpha_1)(x-\alpha_2)\,\phi_2(x)$$

Proceeding in the similar manner we shall get

$$f(x) = (x-\alpha_1)(x-\alpha_2)\cdots\cdots(x-\alpha_{n-1})\,F(x)$$

where $F(x)$ must be a linear factor of the form $a_0(x-\alpha_n)$

$$\therefore \; f(x) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots\cdots(x-\alpha_n)$$

This shows that $f(x)$ is expressed as the product of n linear factors, each factor corresponds to a root and this proves that $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are n roots of the equation $f(x) = 0$

Now we shall prove that there can not be more than n roots.

If possible, let β be a root of the equation $f(x) = 0$ where β is different from $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

Since β is a root, $f(\beta) = 0$ and this would imply

$$a_0(\beta - \alpha_1)(\beta - \alpha_2)(\beta - \alpha_3) \ldots \ldots (\beta - \alpha_n) = 0$$

This is impossible because $a_0 \neq 0$ and

$\beta - \alpha_i \neq 0$ for $i = 1, 2, 3, \ldots\ldots, n$

Therefore $f(x) = 0$ can not have more than

n roots.

In any algebraic equation complex roots occur in conjugate pair.

OR

If $(a+ib)$ be a root of the algebraic equation $f(x)=0$ then $(a-ib)$ is also a root of $f(x)=0$

Proof: Let $(a+ib)$ be a root of the algebraic equation $f(x)=0$

therefore $f(a+ib)=0 \longrightarrow$ ①

We have $\{x-(a+ib)\}\{x-(a-ib)\}$

$$= (x-a)^2 - (ib)^2$$

$$= (x-a)^2 + b^2$$

Let $Q(x)$ and $(Ax+B)$ be the quotient and remainder respectively when $f(x)$ is divided by $\{(x-a)^2+b^2\}$

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$\therefore\ f(x) = \{(x-a)^2+b^2\}\,Q(x) + (Ax+B) \longrightarrow$ (2)

Putting $x = (a+ib)$ in both sides of (2) we get

$$f(a+ib) = 0 + A(a+ib) + B$$

or. $0 = (Aa+B) + ibA$

or $aA + B = 0$ and $bA = 0$ $\longrightarrow$ (3)

i.e $A = 0$ (Since $b \neq 0$)

Putting $A=0$ in (3) we get $B=0$

Therefore from (2) we get

$$f(x) = \{(x-a)^2+b^2\}\,Q(x) \longrightarrow (4)$$

Putting $x = (a-ib)$ in (4) we get

$$f(a-ib) = 0$$

this shows that $(a-ib)$ is also a root of $f(x)=0$

Hence the proof.

Theorem: If $(a+\sqrt{b})$ be a root of the algebraic equation $f(x)=0$, then $(a-\sqrt{b})$ must also be a root of $f(x)=0$

Proof:

Solve $x^4 - 10x^3 + 29x^2 - 22x + 4 = 0$, if one root is $2+\sqrt{3}$

Ans. Since $(2+\sqrt{3})$ is a root of

$$x^4 - 10x^3 + 29x^2 - 22x + 4 = 0 \longrightarrow (1)$$

therefore, $(2-\sqrt{3})$ must also be a root of (1)

Let α and β be other two roots of (1)

therefore we must have

$$\alpha + \beta + (2+\sqrt{3}) + (2-\sqrt{3}) = -\frac{-10}{1} = 10 \longrightarrow (2)$$

$$\Rightarrow \alpha + \beta = 6$$

and $\alpha . \beta . (2+\sqrt{3})(2-\sqrt{3}) = \frac{4}{1}$

$$\Rightarrow \alpha . \beta = 4 \longrightarrow (3)$$

From (2) and (3) it is clear that α and β are the roots of

$$x^2 - 6x + 4 = 0$$

$\therefore x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= 3 \pm \sqrt{5}$

therefore, the roots of ① are $2 \pm \sqrt{3}, 3 \pm \sqrt{5}$

Q Find the roots of the equation $x^4 - 3x^3 - 5x^2 + 9x - 2 = 0$, $(2+\sqrt{3})$ be one root.

Ans: $2 \pm \sqrt{3}$, 1, -2

Q If any one of the root of the equation $x^4 - 9x^3 + 39x^2 - 89x + 78 = 0$ is $(2+3i)$ Find the other roots.

Ans. $2 \pm 3i$, 2, 3

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Prove that the roots of

$$\frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \cdots + \frac{A_n^2}{x-a_n} = 0, x, x-m, k$$

where the constants in the given equation are real, are all real.

OR

the equation can not have any imaginary root.

Proof: $$\frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \cdots\cdots + \frac{A_n^2}{x-a_n} = (x-m) \longrightarrow (1)$$

If possible let, $(\alpha + i\beta)$ be a root of (1) where α and β are real.

Therefore $(\alpha - i\beta)$ must also be a root of (1)

Hence we must have,

$$\frac{A_1^2}{(\alpha+i\beta)-a_1}+\frac{A_2^2}{(\alpha+i\beta)-a_2}+\cdots\cdots+\frac{A_n^2}{(\alpha+i\beta)-a_n}=\left[(\alpha+i\beta)-m\right] \longrightarrow (2)$$

and

$$\frac{A_1^2}{(\alpha-i\beta)-a_1}+\frac{A_2^2}{(\alpha-i\beta)-a_2}+\cdots\cdots+\frac{A_n^2}{(\alpha-i\beta)-a_n}=\left[(\alpha-i\beta)-m\right] \longrightarrow (3)$$

(2) - (3) gives

$$A_1^2\left[\frac{1}{(\alpha-a_1)+i\beta}-\frac{1}{(\alpha-a_1)-i\beta}\right]+A_2^2\left[\frac{1}{(\alpha-a_2)+i\beta}-\frac{1}{(\alpha-a_2)-i\beta}\right]$$

$$+\cdots\cdots\cdots+A_n^2\left[\frac{1}{(\alpha-a_n)+i\beta}-\frac{1}{(\alpha-a_n)-i\beta}\right]=2i\beta$$

or, $$A_1^2\left[\frac{-2i\beta}{(\alpha-a_1)^2+\beta^2}\right]+A_2^2\left[\frac{-2i\beta}{(\alpha-a_2)^2+\beta^2}\right]+\cdots\cdots$$

$$\cdots\cdots\cdots+A_n^2\left[\frac{-2i\beta}{(\alpha-a_n)^2+\beta^2}\right]=2i\beta$$

or, $$-2i\beta\left[\frac{A_1^2}{(\alpha-a_1)^2+\beta^2}+\frac{A_2^2}{(\alpha-a_2)^2+\beta^2}+\cdots+\frac{A_n^2}{(\alpha-a_n)^2+\beta^2}+1\right]=0 \longrightarrow (4)$$

The expression within the third braket on L.H.S of ④ is being possitive, can not be zero.

therefore, $-2i\beta = 0$

$\Rightarrow \beta = 0$

Hence the root of ① are all real.

or, the equation ① can not have any imaginary roots.

Q Prove that

$$\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = x$$

can not have any imaginary roots.

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Prove that the roots of

$$\frac{1}{x+a_1} + \frac{1}{x+a_2} + \cdots\cdots + \frac{1}{x+a_n} = \frac{1}{x}$$

where $a_1, a_2, a_3 \cdots\cdots a_n$ are all positive real numbers, are all real.

Ans. $$\frac{1}{x+a_1} + \frac{1}{x+a_2} + \cdots\cdots + \frac{1}{x+a_n} = \frac{1}{x}$$

$$\Rightarrow \frac{x}{x+a_1} + \frac{x}{x+a_2} + \cdots + \frac{x}{x+a_n} = 1$$

$$\Rightarrow \frac{(x+a_1)-a_1}{x+a_1} + \frac{(x+a_2)-a_2}{x+a_2} + \cdots\cdots + \frac{(x+a_n)-a_n}{x+a_n} = 1$$

$$\Rightarrow \frac{a_1}{x+a_1} + \frac{a_2}{x+a_2} + \cdots\cdots + \frac{a_n}{x+a_n} = n-1 \longrightarrow (1)$$