Errors in Approximate Calculations

Introduction: Numerical analysis is the subject of study and solution of mathematical problems by computation methods. In this context we shall consider various numerical methods for the solution of different mathematical problems and analyze the errors involved in these methods.

Numerical methods give only approximations of the desired true results and, therefore, the study of errors plays a central role in numerical analysis. We now proceed to understand various types of errors.

ERROR

In many cases one has to work with an approximate number, which represents the exact number to certain degree of accuracy.

For example, 0.1111 is an approximate numbers approximating the exact number $\frac{1}{9}$. Obviously, an exact number may have many approximate numbers of different degree of accuracies.

Absolute Error:

If the number x^* is an approximation to the exact number x, then the numerical value of the difference of the exact value and approximated value, i.e.,

$$E_A = |x - x^*|$$

is called absolute error in x^*.

Relative Error:

The relative error in x^* is defined by

$$E_R = \frac{|x - x^*|}{x} = \frac{E_A}{x}$$

Percentage Error: When the relative error expressed in percentage, it is known as percentage error, thus percentage error,

$$E_P = \frac{|x - x^*|}{x} \times 100 = E_R \times 100$$

Approximate numbers

We know that the numbers such as $3, \frac{1}{5}, \frac{1}{8}, 100$ etc are exact numbers because these numbers involve no approximation or uncertaintity.

On the other hand, the number like $\pi, e, \sqrt{3}$ etc. are exact, but can not be expressed exactly by a finite number of digits. They can be written in digital form as $3.1416, 2.7183, 1.7320$ etc. which are only approximations to the true values and are called approximate numbers.

Significant Digits

Significant digits are those, which are used to express a number.

For example, the numbers 1.237, 0.5347, 0.6046 each contains four significant digits but the number 0.0049 has two significant digits since in this case zeros preceeding the non-zero digit serve for fixing the decimal point only.

Rounding off Numbers

It can be easily seen that the number

$$\frac{24}{11.7} = 2.051282051.......$$

never terminates. Thus, for practical computation, it is necessary to cut-off some unnecessary digits and retain only the desired. This is called rounding off numbers.

To round-off a number to n-significant digits, discard all the digits to the right of the nth digit, and if the (n+1)th digit is

(a) less than 5 ; leave the nth digit unchanged,

(b) greater than 5 ; increase the nth digit by 1,

(c) exactly 5 ; increase the nth digit by 1 if it is odd otherwise leave it unchanged.

We cite some examples of rounding off numbers correct to five significant figures:

$5.2908623 \approx 5.2909$

$1.290813 \approx 1.2908$

$31.79992 \approx 31.800$

$4.39065 \approx 4.3906$

$3.48755 \approx 0.4876$

Write following numbers correct upto four significant figures.

$0.00305 \approx 3050 \times 10^{-6}$

$200.51 \approx 200.5$

$630 \approx 6300 \times 10^{-1}$

$0.01020 \approx 1.020 \times 10^{-2}$

$0.0063945 \approx 0.006394$

$0.090038 \approx 0.09004$

Find the sum of the approximate numbers

22.317, 109.0256, 215.006, 36.6, 0.179, 0.0016

Solution: Here 36.6 is the least accurate number and it is correct to one decimal place. Hence we round off other numbers upto two decimal places to get the required sum as

$$22.32 + 109.02 + 215.01 + 36.6 + 0.18 + 0.00 = 383.13$$
$$\simeq 383.1$$

Subtract 203.176 from 791.23

Solution: Rounding off 203.176 to two decimal places given by 203.18, we get the required difference as

$$791.23 - 203.18 = 588.05$$

Find the product of the approximate numbers 1.237, 209.188, 78.6, 25.006

Solution: Since the least accurate number is 78.6, we round off the given numbers to two decimal places and so we get the required product as

$$1.24 \times 209.19 \times 78.6 \times 25.01 = 509916.24$$
$$\simeq 509916.2$$

Find the absolute, relative and percentage error when $\frac{3}{7}$ is approximated by 0.4286

Solution:

The absolute error is given by

$$E_A = \left| \frac{3}{7} - 0.4286 \right|$$

$$= 2.857 \times 10^{-5}$$

The relative error is given by

$$E_R = \frac{E_A}{\frac{3}{7}} = 6.666 \times 10^{-5}$$

Thus the percentage error is given by

$$E_P = E_R \times 100$$

$$= 6.666 \times 10^{-3}$$

Write down approximate representation of $\frac{2}{3}$ correct to five significant figures and then find absolute, relative and percentage errors.

Solution: $\frac{2}{3} = 0.66667$

Now true value $= x_T = \frac{2}{3}$

Approximate value $= x_A = 0.66667$

Hence absolute error $= |x_T - x_A|$

$= \left| \frac{2}{3} - 0.66667 \right|$

$= 0.0000033$

Also relative error $= E_R = \frac{0.0000033}{\frac{2}{3}}$

$= 0.00000495$

$\simeq 0.000005$

$\therefore$ percentage error $= E_R \times 100$

$= 0.0005\%$

If $\pi = 3.14$ in place of 3.14156, find the relative error.

Solution: Here true value $x_T = 3.14156$

Approximate value $x_A = 3.14$

Absolute Error $= E_A = |x_T - x_A|$

$= |3.14156 - 3.14|$

$= 0.00156$

$\therefore$ Relative Error $= \dfrac{E_A}{x_T} = \dfrac{0.00156}{3.14156}$

$= 0.000496$