Elementary theory of Probability

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Sample Space

- The **sample space** is the **set of all possible outcomes** of a random experiment.
- Denoted by S
- **Definition (Set form):**
- $S = \{all\ possible\ outcomes\}$
- **Example:**
  If a coin is tossed,
- $S = \{H, T\}$
- where H= Head, T= Tail
- If a die is rolled,
- $S = \{1, 2, 3, 4, 5, 6\}$

Classical Definition of Probability

- The **Classical Probability** applies when all outcomes are **equally likely**.
- **Definition**
- If a random experiment has n equally likely outcomes and m favorable outcomes for event A, then:
- $P(A) = \frac{m}{n}$
- **Example**
- Rolling a fair die:
- Sample space: $S = \{1,2,3,4,5,6\}$
- Event A: getting an even number $= \{2,4,6\}$
- $P(A) = \frac{3}{6} = \frac{1}{2}$

Event

- An **event** is any **subset of the sample space**.
- Denoted by capital letters like A, B, E
- **Definition (Set form):**
- $A \subseteq S$
-

Types of Events (Using Set Theory)

- **Simple Event**

Contains only one outcome
Example: $A = \{3\}$

Events _ Contd

Compound Event

Contains more than one outcome
Example:

$$A = \{2,4,6\} \ \ (\textit{even numbers})$$

Sure Event

Event that always occurs

$$S \ \ (\textit{the whole sample space})$$

Impossible Event

Event that never occurs

$$\varnothing \ \ (\textit{empty set})$$

Axiomatic Definition of Probability

- The **Axiomatic Probability**, developed by Andrey Kolmogorov, is the **modern and most general approach**.

Definition

- Probability is a function Pdefined on events satisfying:
- **Axioms**
- **Non-negativity**
- $P(A) \geq 0$
- **Normalization**
- $P(S) = 1$
- **Additivity (for mutually exclusive events)**
- $P(A \cup B) = P(A) + P(B), \, if \, A \cap B = \varnothing$

Comparison

Basis	Classical Definition	Axiomatic Definition
Nature	Simple, intuitive	Abstract, mathematical
Assumption	Outcomes must be equally likely	No such assumption
Applicability	Limited (games of chance)	Very general (all probability models)
Foundation	Counting favorable cases	Based on axioms
Developed by	Early mathematicians	Kolmogorov

Theorem of Total Probability

- The **Law of Total Probability** helps us find the probability of an event by breaking it into simpler parts.
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Statement

- If $B_1, B_2, \ldots, B_n$ are **mutually exclusive and exhaustive events** (i.e., they form a partition of the sample space S), and A is any event, then:
- $P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$
-

Conditions

- $B_1, B_2, \ldots, B_n$ are **disjoint**: $B_i \cap B_j = \varnothing$ for $i \neq j$
- They are **exhaustive**: $B_1 \cup B_2 \cup \cdots \cup B_n = S$
- $P(B_i) > 0$
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Intuition (Simple Idea)

- You split the event A into parts depending on different cases B_i, then add their probabilities.
- "Total probability = sum of conditional probabilities × their weights"

Theorem of Total Probability _ Contd.

Example

- Suppose:
- A factory has 2 machines:
  - B_1: Machine 1 produces 60% items → $P(B_1) = 0.6$
  - B_2: Machine 2 produces 40% items → $P(B_2) = 0.4$
- Defective rates:
  - $P(A \mid B_1) = 0.02$
  - $P(A \mid B_2) = 0.05$
- Find probability that a product is defective $P(A)$.

✔**Solution**

- $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$
- $= (0.02)(0.6) + (0.05)(0.4)$
- $= 0.012 + 0.020 = 0.032$

Theorem of Compound Probability

- The **Multiplication Rule of Probability** (also called the theorem of compound probability) gives the probability that **two or more events occur together**.
- $P(A \cap B) = P(A)P(B|A)$
- $P(A \cap B) = P(B)P(A|B)$

Example:

- P(A)
- $P(B \mid A)$
- $P(A \cap B) = P(A) \cdot P(B \mid A) \approx 0.21$

Special Case: Independent Events

- If A and B are **independent**, then:
- $P(B \mid A) = P(B)$
- So,
- $P(A \cap B) = P(A)\, P(B)$

Extension (Three Events)

- For three events A, B, C:
- $P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Example of Compound Probability

- A bag contains 3 red and 2 blue balls. Two balls are drawn **without replacement**.
- Let:
- A: first ball is red
- B: second ball is red
- ✔ **Solution**
- $P(A) = \frac{3}{5}$
- After one red is removed, remaining red balls = 2 out of 4:
- $P(B \mid A) = \frac{2}{4} = \frac{1}{2}$
- Using compound probability:
- $P(A \cap B) = P(A)\ P(B \mid A) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$

Bayes' Theorem

- The **Bayes' Theorem** tells us how to **update probabilities** when new information is available.
- **Statement**
- If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events and A is an event with $P(A) > 0$, then:
- $P(B_i \mid A) = \frac{P(A|B_i)\,P(B_i)}{\sum_{j=1}^{n} P(A|B_j)\,P(B_j)}$

Contd.

- **Meaning (Simple Idea)**

- It helps us find the **probability of a cause given an observed result**.
- $P(B_i)\to$ prior probability (before knowing A)
- $P(A \mid B_i)\to$ likelihood
- $P(B_i \mid A)\to$ posterior probability (after knowing A)

Example

Suppose:

- Two factories produce goods:
  - B_1: Factory 1 → 70% production
  - B_2: Factory 2 → 30% production
- Defective rates:
  - $P(A \mid B_1) = 0.01$
  - $P(A \mid B_2) = 0.03$

A product is found defective. Find probability it came from Factory 2.

Contd.

- **Step 1: Total Probability**
- $P(A) = (0.01)(0.7) + (0.03)(0.3) = 0.007 + 0.009 = 0.016$
- **Step 2: Apply Bayes' Theorem**
- $P(B_2 \mid A) = \frac{P(A|B_2)P(B_2)}{P(A)} = \frac{0.03 \times 0.3}{0.016} = \frac{0.009}{0.016} = 0.5625$
- **Final Answer**
- $P(B_2 \mid A) = 0.5625$