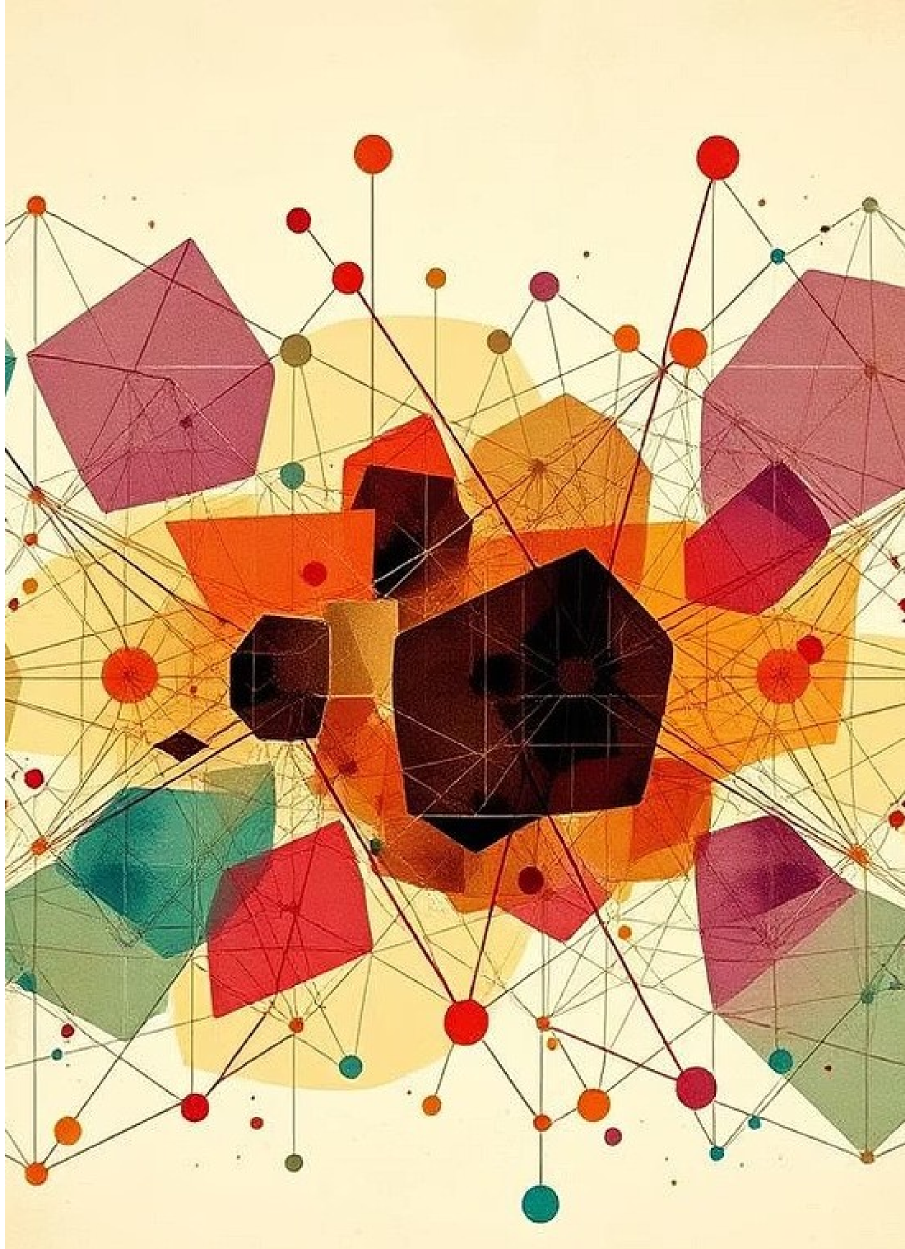
Introduction Matrices



Elementary Matrix Operations

Matrix Addition

Matrices of the same dimensions can be added by summing their corresponding elements. This operation is only defined for matrices with an identical number of rows and columns.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

Matrix Subtraction

Similar to addition, matrices of the same dimensions can be subtracted.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} 2 \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \, 2 \, e & b \, 2 \, f \\ c \, 2 \, g & d \, 2 \, h \end{bmatrix}$$

Properties of Matrix Addition

Commutative

$A + B = B + A$

The order of addition does not affect the result.

Associative

$(A + B) + C = A + (B + C)$

Grouping of matrices does not affect the sum.

Rank of Matrices

Definition

Maximum number of linearly independent rows or columns in a matrix.

How to Find

Row-reduce the matrix to its echelon form, then count the number of non-zero rows.

Properties

- $\text{rank}(A) \leq \min(\text{rows}, \text{columns})$.
- Full rank occurs when $\text{rank}(A) = \min(\text{rows}, \text{columns})$.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \text{rank} = 1$$

(Rows are linearly dependent)

Determinants of a Square Matrix

Definition

A determinant is a **scalar value** computed from the elements of a square matrix. It provides important information about the matrix, such as its invertibility.

2x2 Formula

For a 2x2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad \; 2 \; bc$$

Key Properties

$\det(I) = 1, \det(\text{zero matrix}) = 0$

Swapping rows reverses the sign of the determinant.

$\det(AB) = \det(A)\det(B)$

Inverse of a Square Matrix

Definition

A^{-1} such that $AA^{-1} = I$. Exists if $\det(A) \neq 0$.

Calculation Methods

Adjugate Formula: $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$. Or Row Reduction: $[A|I] \to [I|A^{-1}]$.

Properties

$(AB)^{-1} = B^{-1}A^{-1}$. $(A^{-1})^T = (A^T)^{-1}$.

Example

For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $\det(A) = 5$, then $A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$.

Solution of a System of Linear Equations using **Cramer's Rule**

Cramer’s Rule is a method to solve a system of linear equations using determinants. It works only when the number of equations equals the number of unknowns and the determinant of the coefficient matrix is **non-zero**.

Consider the system:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Step 1: Compute the main determinant (D)

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Step 2: Compute D_x and D_y

Replace columns with constants:

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Step 3: Find the solution

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}$$

Example

2x+3y=8

$$x + 2y = 5$$

Step 1: Determinant D

$$D = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (2 \times 2 - 1 \times 3) = 4 - 3 = 1$$

Step 2: Compute D_x, D_y

$$D_x = \begin{vmatrix} 8 & 3 \\ 5 & 2 \end{vmatrix} = (8 \times 2 - 5 \times 3) = 16 - 15 = 1$$

$$D_y = \begin{vmatrix} 2 & 8 \\ 1 & 5 \end{vmatrix} = (2 \times 5 - 1 \times 8) = 10 - 8 = 2$$

Step 3: Final Answer

$$x = \frac{1}{1} = 1, \; y = \frac{2}{1} = 2$$

Eigenvalues and Eigenvectors

Eigenvector
A **non-zero vector** X is called an eigenvector of matrix A if, when multiplied by A, it only changes by a scalar factor (not direction).

Eigenvalue
The scalar λ corresponding to that eigenvector is called the eigenvalue.

Mathematical Definition

$$AX = \lambda X$$

Where:
A= square matrix
$X \neq 0$= eigenvector
λ= eigenvalue

In Simple Words
Eigenvector → direction remains same
Eigenvalue → amount of stretching or shrinking
Characteristic equation: $|A-\lambda I|=0$

Example

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 1: Characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

☞ **Eigenvalues:**

$$\lambda_1 = 3, \lambda_2 = 1$$

For $\lambda = 3$

Solve:

$$(A-3I)X = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

☞ Equation: $-x + y = 0 \rightarrow x = y$

✔ Eigenvector:

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example _ contd.

For $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

☞ Equation: $x + y = 0 \rightarrow x = -y$

✔ Eigenvector:

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

•Eigenvectors are **not unique** (any scalar multiple is also valid).

•Used in:

Economics (input-output models)

Data science (PCA)

Differential equations

Thank You!

We appreciate your attention and hope this presentation provided valuable insights into Set Theory and Matrices.